

Character expansion for HOMFLY polynomials.

III. All 3-Strand braids in the first symmetric representation

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We continue the program of systematic study of extended HOMFLY polynomials, suggested in [1, 2]. Extended polynomials depend on infinitely many time-variables, are close relatives of integrable τ -functions, and depend on the choice of the braid representation of the knot. They possess natural character decompositions, with coefficients which can be defined by exhaustively general formula for any particular number m of strands in the braid and any particular representation R of the Lie algebra $GL(\infty)$. Being restricted to "the topological locus" in the space of time-variables, the extended HOMFLY polynomials reproduce the ordinary knot invariants. We derive such a general formula, for $m = 3$, when the braid is parameterized by a sequence of integers $(a_1, b_1, a_2, b_2, \dots)$, and for the first non-fundamental representation $R = [2]$. Instead of calculating the mixing matrices directly, as suggested in [2], we deduce them from comparison with the known answers for torus and composite knots. A simple reflection symmetry converts the answer for the symmetric representation [2] into that for the antisymmetric one [1, 1]. The result applies, in particular, to the figure eight knot 4_1 , and was further extended to superpolynomials in arbitrary symmetric and antisymmetric representations in [3].

1 Introduction

In [1, 2] we started a program to promote the HOMFLY polynomials [4] to character expansions, representing them as linear combinations of the Schur functions $S_Q\{p_k\}$ (i.e. the characters of linear groups) [5]. Such an expansion depends on the choice of a braid realization of the knot, thus, its coefficients by themselves are not knot invariants, instead they are pure group theory quantities and possess many nice properties. For an m -strand braid \mathcal{B} the HOMFLY polynomial in representation R is expanded as¹

$$H_R^{\mathcal{B}} = \text{Tr}_{R^{\otimes m}}((q^\rho)^{\otimes m} \mathcal{B}) = \sum_{Q \vdash m|R} h_{RQ}^{\mathcal{B}} S_Q^*(A) \quad (1)$$

where

$$S_Q^*(A) = \text{Tr}_{Q \in R^{\otimes m}}(q^\rho)^{\otimes m} = S_Q\{p_k^*\}, \quad p_k^* = \frac{[kN]}{[k]} = \frac{A^k - A^{-k}}{q^k - q^{-k}} \quad (2)$$

are quantum dimensions of representations Q of $SU(N)$, where $A = q^N$ and $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$. The coefficients $h_{RQ}^{\mathcal{B}}$ do not depend on A , i.e. on N , thus, they can be evaluated from analysis of an arbitrary group $SU_q(N)$.

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¹Our calculus is based on the approach by [6], though that of [7, 8, 9] is, by essence, also very close. The first part of this formula is related to Chern-Simons theory in the temporal gauge [10]. The new element is a special emphasis on the character expansion, which allows one to extend the knot polynomials to arbitrary time variables [1] and provides very simple *general* formulas for entire classes of knots. However, following this line, we omit the additional *factor* $(A^\alpha q^\gamma)^{w^{\mathcal{B}}}$ in front on the trace in (1), where $w^{\mathcal{B}}$ is the writhe number of the braid, while α and γ depend on normalization of the \mathcal{R} -matrix. Throughout this paper we use a special normalization of \mathcal{R} -matrices, though in our normalization α and γ are actually non-vanishing: for most purposes (in the standard framing) $\alpha = -|R|$ and $\gamma = -4\kappa_R$. We discuss this issue in a separate subsection 3.11, and actually restore the factors in the formulas of the Appendix. To simplify the notations we do not put star on $H(A)$ in this paper, as we do in [1, 2, 3], because the extended polynomials are almost not mentioned here. They can be, however, obtained from the formulas of the Appendix simply by removing the stars from $*S_Q$, thus, promoting them from the quantum dimensions to the Schur functions.

Instead they can be represented as traces in auxiliary spaces of intertwining operators $\mathcal{M}_{R^m Q}$, whose dimension is the number $\dim \mathcal{M}_{R^m Q} = N_{R^m Q}$ of times the irreducible representation Q appears in the m -th tensor power of representation R ,

$$R^{\otimes m} = \sum_{Q \vdash m|R|} \mathcal{M}_{R^m Q} \otimes Q \quad (3)$$

The trace is taken of a product of the diagonal quantum \mathcal{R} -matrices $\hat{\mathcal{R}}$ acting in $\mathcal{M}_{R^m Q}$, and the "mixing matrices" intertwining \mathcal{R} -matrices that act on different pairs of adjacent strands in the braid. These mixing matrices, in their turn, can be represented as products of universal constituents, associated with a switch between two "adjacent" trees describing various decompositions (3).

In [2] we exhaustively described such representations for the coefficients $h_{RQ}^{\mathcal{B}}$ for *arbitrary* $m = 2, 3, 4$ -strand braids and for the simplest fundamental representation $R = [1]$:

$$m = 2, \quad \mathcal{B} = \mathcal{R}^a : \quad H_{[1]}^{(a)} = q^a S_2^*(A) + \left(-\frac{1}{q}\right)^a S_{11}^*(A) = q^a S_2^*(A) + \left(q \rightarrow -\frac{1}{q}\right) \quad (4)$$

$$m = 3, \quad \mathcal{B} = (\mathcal{R} \otimes I)^{a_1} (I \otimes \mathcal{R})^{b_1} (\mathcal{R} \otimes I)^{a_2} (I \otimes \mathcal{R})^{b_2} \dots \quad (5)$$

$$H_{[1]}^{(a_1, b_1 | a_2, b_2 | \dots)} = q^{\sum_i (a_i + b_i)} S_3^*(A) + \left(-\frac{1}{q}\right)^{\sum_i (a_i + b_i)} S_{111}^*(A) + \left(\text{Tr}_{2 \times 2} \hat{\mathcal{R}}_2^{a_1} U_2 \hat{\mathcal{R}}_2^{b_1} U_2^\dagger \hat{\mathcal{R}}_2^{a_2} U_2 \hat{\mathcal{R}}_2^{b_2} U_2^\dagger \dots\right) S_{21}^*(A)$$

$$m = 4, \quad \mathcal{B} = (\mathcal{R} \otimes I \otimes I)^{a_1} (I \otimes \mathcal{R} \otimes I)^{b_1} (\mathcal{R} \otimes I \otimes I)^{c_1} (\mathcal{R} \otimes I \otimes I)^{a_2} (I \otimes \mathcal{R} \otimes I)^{b_2} (\mathcal{R} \otimes I \otimes I)^{c_2} \dots \quad (6)$$

$$\begin{aligned} H_{[1]}^{(a_1, b_1, c_1 | a_2, b_2, c_2 | \dots)} &= q^{\sum_i (a_i + b_i + c_i)} S_4^*(A) + \left(-\frac{1}{q}\right)^{\sum_i (a_i + b_i + c_i)} S_{1111}^*(A) + \\ &+ \left(\text{Tr}_{2 \times 2} \hat{\mathcal{R}}_2^{a_1} U_2 \hat{\mathcal{R}}_2^{b_1} U_2^\dagger \hat{\mathcal{R}}_2^{c_1 + a_2} U_2 \hat{\mathcal{R}}_2^{b_2} U_2^\dagger \hat{\mathcal{R}}_2^{c_2 + a_3} \dots\right) S_{22}^*(A) + \\ &+ \left\{ \left(\text{Tr}_{3 \times 3} \hat{\mathcal{R}}_3^{a_1} U_3 \hat{\mathcal{R}}_3^{b_1} V_3 U_3 \hat{\mathcal{R}}_3^{c_1} U_3^\dagger V_3^\dagger U_3^\dagger \hat{\mathcal{R}}_3^{a_2} U_3 \hat{\mathcal{R}}_3^{b_2} V_3 U_3 \hat{\mathcal{R}}_3^{c_2} U_3^\dagger V_3^\dagger U_3^\dagger \dots\right) S_{31}^*(A) + \left(q \rightarrow -\frac{1}{q}\right) \right\} \end{aligned}$$

where

$$\begin{aligned} \hat{\mathcal{R}}_2 &= \begin{pmatrix} q & \\ & -\frac{1}{q} \end{pmatrix} & \hat{\mathcal{R}}_3 &= \begin{pmatrix} q & & \\ & q & \\ & & -\frac{1}{q} \end{pmatrix} \\ U_2 &= \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix} & U_3 &= \begin{pmatrix} 1 & & \\ & c_2 & s_2 \\ & -s_2 & c_2 \end{pmatrix} & V_3 &= \begin{pmatrix} c_3 & s_3 \\ -s_3 & c_3 \\ & & 1 \end{pmatrix} \end{aligned} \quad (7)$$

Subscripts refer to the size of the matrices, the entries of rotation matrices U and V are given by

$$c_k = \frac{1}{[k]}, \quad s_k = \sqrt{1 - c_k^2} = \frac{\sqrt{[k-1][k+1]}}{[k]} \quad (8)$$

These formulas provide a very transparent and convenient representation for infinitely many HOMFLY polynomials and seem to be very useful for any theoretical analysis of their general properties, from integrability to linear Virasoro like relations (including A -polynomials and spectral curves [11, 12, 13, 14], AMM/EO topological recursion [15, 16] etc). Therefore, further insights are important about the structure of these formulas and their generalizations (in [2] the $m = 5$ case was also investigated, and the general formula for the coefficients $h_{[1]}^{[m-1, 1]}$ was suggested for all m).

The limitations in [2] are pure technical: to make the paper readable and the main ideas understandable, we considered only the implications of the theory of $SU_q(2)$ quantum group: this allowed us to calculate only

contributions of the Young diagrams Q with one and two columns or rows. Including the diagrams with l columns or rows, which arise when the number of strands in the braid is $m \geq 5$, requires the similar use of the $SU_q(l)$ quantum group theory, which is tedious but straightforward, and will be considered in further publications. (We emphasize once again that l has nothing to do with N in $A = q^N$, the relevant l is related to the number m of strands, and for small $m \leq 4$ the smallest $l = 2$ is sufficient to describe *all* the HOMFLY polynomials with $R = [1]$.)

Another restriction in [2] was to $R = [1]$. It is also partly related to restriction to $SU_q(2)$, but not only. It is the purpose of the present paper to make a first step in the direction towards the *colored* HOMFLY polynomials with $|R| > 1$, that is, to the symmetric representation $R = [2]$. **Instead of performing this calculation directly, *a la* [2], we use a shortcut: we determine the five parameters (angles) of the three orthogonal matrices $\hat{U}_{[51]}$, $\hat{U}_{[321]}$ and $\hat{U}_{[42]}$ by comparison with the answers for torus and composite knots and links in eq.(24) below.** This can be also compared with the results of the long-lasting impressive work by the Indian group [7] using the direct evaluation of the Racah coefficients.

Our goal is to find the necessary ingredients for formulas like (4)-(6), which provide an *exhaustive* description of *all* braids with a given number of strands and in a given representation. The final task would be to find general formulas that depend explicitly on *all* the parameters: the number of strands m , the set (a_{1i}, \dots, a_{mi}) , specifying the m -strand braid (i.e. on the element of the braid group), and on the Young diagram R labeling the representation. The formula is going to be for a coefficient in front of a particular character, like the Schur function $S_Q\{p\}$ (or, alternatively, Hall-Littlewood [17] or some other element of an appropriate expansion basis). Now we perform a step of this program: find such *parametric* formulas for given $m = 3$ and $R = [2]$. For $R = [1]$ and $m = 3, 4$ (and partly $m = 5$) they were already derived in [2].

The formulas of this paper for the mixing matrices, which we obtain here indirectly, can be obtained directly using the representation theory, like it was done in [2]. We consider the mixing matrices for a more generic case in [18].

2 The case of 2 strands, $m = 2$

To determine emerging Q in this case, one suffices to expand the product of two symmetric representations:

$$[2] \times [2] = [4] + [31] + [22], \quad [11] \times [11] = [22] + [211] + [1111] \quad (9)$$

This decomposition can be easily obtained from the decomposition of the characters. Indeed, given $S_{[2]} = \frac{1}{2}p_2 + \frac{1}{2}p_1^2$, $S_{[11]} = -\frac{1}{2}p_2 + \frac{1}{2}p_1^2$ and

$$\begin{aligned} S_{[4]} &= \frac{1}{4}p_4 + \frac{1}{3}p_3p_1 + \frac{1}{8}p_2^2 + \frac{1}{4}p_2p_1^2 + \frac{1}{24}p_1^4, \\ S_{[31]} &= -\frac{1}{4}p_4 - \frac{1}{8}p_2^2 + \frac{1}{4}p_2p_1^2 + \frac{1}{8}p_1^4, \\ S_{[22]} &= -\frac{1}{3}p_3p_1 + \frac{1}{4}p_2^2 + \frac{1}{12}p_1^4, \\ S_{[211]} &= \frac{1}{4}p_4 - \frac{1}{8}p_2^2 - \frac{1}{4}p_2p_1^2 + \frac{1}{8}p_1^4, \\ S_{[1111]} &= -\frac{1}{4}p_4 + \frac{1}{3}p_3p_1 + \frac{1}{8}p_2^2 - \frac{1}{4}p_2p_1^2 + \frac{1}{24}p_1^4, \end{aligned} \quad (10)$$

it is easy to check that

$$S_{[2]}^2 = S_{[4]} + S_{[31]} + S_{[22]}, \quad S_{[11]}^2 = S_{[22]} + S_{[211]} + S_{[1111]} \quad (11)$$

In particular, for the ordinary dimensions $d_Q = S_Q\{p_k = N\}$ of representations Q of the $SU(N)$ algebra it reads as $\left(\frac{N(N+1)}{2}\right)^2 = \frac{N(N+1)(N+2)(N+3)}{24} + \frac{(N-1)N(N+1)(N+2)}{8} + \frac{(N^2-1)N^2}{12}$ etc. and very similarly for the quantum dimensions $D_Q = S_Q^* = S_Q\{p_k = [N]_q\}$:

$$\begin{aligned} \left(\frac{[N][N+1]}{[2]}\right)^2 &= \frac{[N][N+1][N+2][N+3]}{[2][3][4]} + \frac{[N-1][N][N+1][N+2]}{[2][4]} + \frac{[N-1][N]^2[N+1]}{[2]^2[3]} \\ \left(\frac{[N-1][N]}{[2]}\right)^2 &= \frac{[N-1][N]^2[N+1]}{[2]^2[3]} + \frac{[N-2][N-1][N][N+1]}{[2][4]} + \frac{[N-3][N-2][N-1][N]}{[2][3][4]} \end{aligned} \quad (12)$$

The HOMFLY polynomials are given by [19] (see also [20] for an extension to superpolynomials [21])

$$\begin{aligned}
H_{[1]}^{2,n} &= \sum_{Q \in [1] \times [1]} \epsilon_Q^n q^{n\kappa_Q} S_Q = q^n S_{[2]} \mp q^{-n} S_{[11]} \\
H_{[2]}^{2,n} &= \sum_{Q \in [2] \times [2]} \epsilon_Q^n q^{n\kappa_Q} S_Q = q^{6n} S_{[4]} \mp q^{2n} S_{[31]} + S_{[22]} \\
H_{[11]}^{2,n} &= \sum_{Q \in [2] \times [2]} \epsilon_Q^n q^{n\kappa_Q} S_Q = q^{-6n} S_{[1111]} \mp q^{-2n} S_{[211]} + S_{[22]}
\end{aligned} \tag{13}$$

where² $\kappa_Q = -\nu_Q + \nu_{Q'}$, $\nu_Q = \sum (i-1)Q_i$, Q' is the transposed Young diagram and the sign factors ϵ_Q are defined from the Adams rule [19]: "the initial condition" $H_{[R]}^{m,0}\{p\} = \widehat{Ad}_m S_R\{p\}$ with $\widehat{Ad}_m p_k = p_{mk}$ is imposed at $n = 0$

$$\begin{aligned}
\widehat{Ad}_2 S_{[1]}\{p\} &= \frac{p_2 + p_1^2}{2} = S_{[2]} - S_{[11]} \\
\widehat{Ad}_2 S_{[2]}\{p\} &= \frac{p_4 + p_2^2}{2} = S_{[4]} - S_{[31]} + S_{[22]} \\
\widehat{Ad}_2 S_{[11]}\{p\} &= \frac{p_4 + p_2^2}{2} = S_{[1111]} - S_{[211]} + S_{[22]}
\end{aligned} \tag{14}$$

It should not be mixed with the "physical" initial conditions for the n -evolution of [20],

$$H_R^{m,n} = H_R^{n,m} \tag{15}$$

imposed at $0 < n < m$.

For links one has instead of (14):

$$\begin{aligned}
\left(\widehat{Ad}_1 S_{[1]}\{p\}\right)^2 &= p_1^2 = S_{[2]} + S_{[11]}, \\
\left(\widehat{Ad}_1 S_{[2]}\{p\}\right)^2 &= \frac{(p_2 + p_1^2)^2}{4} = S_{[4]} + S_{[31]} + S_{[22]}, \\
\left(\widehat{Ad}_1 S_{[11]}\{p\}\right)^2 &= \frac{(-p_2 + p_1^2)^2}{4} = S_{[1111]} + S_{[211]} + S_{[22]}
\end{aligned} \tag{16}$$

Accordingly, the signs \mp at the r.h.s. of (13) are minuses and pluses for knots and links (n odd or even) respectively.

In particular, for the unknot with $(m, n) = (2, 1)$

$$\begin{aligned}
H_{[1]}^{2,1} &= q S_{[2]}^* - q^{-1} S_{[11]}^* = q S_{[1]}^* \\
H_{[2]}^{2,1} &= q^6 S_{[4]}^* - q^2 S_{[31]}^* + S_{[22]}^* = A^2 q^4 S_{[2]}^* \\
H_{[11]}^{2,1} &= q^{-6} S_{[1111]}^* - q^{-2} S_{[211]}^* + S_{[22]}^* = A^2 q^4 S_{[11]}^*
\end{aligned} \tag{17}$$

for the Hopf link with $(m, n) = (2, 2)$

$$\begin{aligned}
H_{[1]}^{2,2} &= q^2 S_{[2]}^* + q^{-2} S_{[11]}^* = ((q^2 - 1 + q^{-2})A - A^{-1}) \frac{S_{[1]}^*}{\{q\}} \\
H_{[2]}^{2,2} &= q^{12} S_{[4]}^* + q^4 S_{[31]}^* + S_{[22]}^* = ((q^{12} - q^{10} - q^8 + 2q^6 - q^2 + 1)A^2 - q^8 + q^4 - q^2 - 1 + q^2 A^{-2}) q \frac{S_{[2]}^*}{\{q\}\{q^2\}} \\
H_{[11]}^{2,2} &= q^{-12} S_{[1111]}^* + q^{-4} S_{[211]}^* + S_{[22]}^* = \\
&= ((1 - q^{-2} + 2q^{-6} - q^{-8} - q^{-10} + q^{-12})A^2 - 1 - q^{-2} + q^{-4} - q^{-8} + q^{-2} A^{-2}) \frac{S_{[11]}^*}{q\{q\}\{q^2\}}
\end{aligned} \tag{18}$$

²Similarly to [1, 2], we use here $\kappa_Q = -\nu_Q + \nu_{Q'}$ with the opposite sign as compared with [20].

and for the trefoil with $(m, n) = (2, 3)$

$$\begin{aligned}
H_{[1]}^{2,3} &= q^3 S_{[2]}^* - q^{-3} S_{[11]}^* = ((q^2 + q^{-2})A - A^{-1}) S_{[1]}^* \\
H_{[2]}^{2,3} &= q^{18} S_{[4]}^* - q^6 S_{[31]}^* + S_{[22]}^* = ((q^{12} + q^6 + q^4 + 1)A^2 - q^8 - q^6 - q^2 - 1 + q^2 A^{-2}) q^4 S_{[2]}^* \\
H_{[11]}^{2,3} &= q^{-18} S_{[1111]}^* - q^{-6} S_{[211]}^* + S_{[22]}^* = ((q^{-12} + q^{-6} + q^{-4} + 1)A^2 - q^{-8} - q^{-6} - q^{-2} - 1 + q^{-2} A^{-2}) \frac{S_{[11]}^*}{q^4}
\end{aligned} \tag{19}$$

3 3 strands, $m = 3$

3.1 Structure of the answer

Now

$$\begin{aligned}
[2] \times [2] \times [2] &= ([4] + [31] + [22]) \times [2] = \\
&= ([6] + [51] + [42]) + ([51] + [42] + [411] + [33] + [321]) + ([42] + [321] + [222])
\end{aligned} \tag{20}$$

For example, for the dimensions of $SU(2)$ representations one has $3^3 = 27 = (7 + 5 + 3) + (5 + 3 + 0 + 1 + 0) + (3 + 0 + 0)$. Again, this decomposition is obtained as the decomposition of the characters:

$$\begin{aligned}
S_{[6]} &= \frac{1}{6}p_6 + \frac{1}{5}p_5p_1 + \frac{1}{8}p_4p_2 + \frac{1}{8}p_4p_1^2 + \frac{1}{18}p_3^2 + \frac{1}{6}p_3p_2p_1 + \frac{1}{18}p_3p_1^3 + \frac{1}{48}p_2^3 + \frac{1}{16}p_2^2p_1^2 + \frac{1}{48}p_2p_1^4 + \frac{1}{720}p_1^6, \\
S_{[51]} &= -\frac{1}{6}p_6 - \frac{1}{8}p_4p_2 + \frac{1}{8}p_4p_1^2 - \frac{1}{18}p_3^2 + \frac{1}{9}p_3p_1^3 - \frac{1}{48}p_2^3 + \frac{1}{16}p_2^2p_1^2 + \frac{1}{16}p_2p_1^4 + \frac{1}{144}p_1^6, \\
S_{[42]} &= -\frac{1}{5}p_5p_1 + \frac{1}{8}p_4p_2 - \frac{1}{8}p_4p_1^2 + \frac{1}{16}p_3^2 + \frac{1}{16}p_2^2p_1^2 + \frac{1}{16}p_2p_1^4 + \frac{1}{80}p_1^6, \\
S_{[411]} &= \frac{1}{6}p_6 + \frac{1}{18}p_3^2 - \frac{1}{6}p_3p_2p_1 + \frac{1}{18}p_3p_1^3 - \frac{1}{24}p_2^3 - \frac{1}{8}p_2^2p_1^2 + \frac{1}{24}p_2p_1^4 + \frac{1}{72}p_1^6, \\
S_{[33]} &= -\frac{1}{8}p_4p_2 - \frac{1}{8}p_4p_1^2 + \frac{1}{9}p_3^2 + \frac{1}{6}p_3p_2p_1 - \frac{1}{18}p_3p_1^3 - \frac{1}{16}p_2^3 + \frac{1}{16}p_2^2p_1^2 + \frac{1}{48}p_2p_1^4 + \frac{1}{144}p_1^6, \\
S_{[321]} &= \frac{1}{5}p_5p_1 - \frac{1}{9}p_3^2 - \frac{1}{9}p_3p_1^3 + \frac{1}{45}p_1^6, \\
&\dots \\
S_{[222]} &= -\frac{1}{8}p_4p_2 + \frac{1}{8}p_4p_1^2 + \frac{1}{9}p_3^2 - \frac{1}{6}p_3p_2p_1 - \frac{1}{18}p_3p_1^3 + \frac{1}{16}p_2^3 + \frac{1}{16}p_2^2p_1^2 - \frac{1}{48}p_2p_1^4 + \frac{1}{144}p_1^6, \\
&\dots
\end{aligned} \tag{21}$$

Thus, one needs the 2×2 mixing matrices for representations $[51]$ and $[321]$ and the 3×3 mixing matrix for representation $[42]$.

The answer for the HOMFLY polynomial in the fundamental representation for the generic 3-strand knot $(a_1, b_1 | a_2, b_2 | \dots)$ has the following form:

$$\begin{aligned}
H_{[1]}^{a_1, b_1 | a_2, b_2 | \dots} &= q^{a_1 + b_1 + a_2 + b_2 + \dots} S_{[3]} + \left(-\frac{1}{q}\right)^{a_1 + b_1 + a_2 + b_2 + \dots} S_{[111]} + \\
&+ \text{tr}_{2 \times 2} \left\{ \hat{\mathcal{R}}_{[21]}^{a_1} \hat{U}_{[21]} \hat{\mathcal{R}}_{[21]}^{b_1} \hat{U}_{[21]}^\dagger \hat{\mathcal{R}}_{[21]}^{a_2} \hat{U}_{[21]} \hat{\mathcal{R}}_{[21]}^{b_2} \hat{U}_{[21]}^\dagger \dots \right\} S_{[21]}
\end{aligned} \tag{22}$$

with

$$\hat{\mathcal{R}}_{[21]} = \begin{pmatrix} q^{\mathcal{R}_{[2]}} & 0 \\ 0 & -q^{\mathcal{R}_{[11]}} \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, \quad \hat{U}_{[21]} = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix} \tag{23}$$

Likewise, in the symmetric representation, it is going to be

$$\begin{aligned}
H_{[2]}^{a_1, b_1 | a_2, b_2 | \dots} &= (q^6)^{a_1+b_1+a_2+b_2+\dots} S_{[6]} + (-q^2)^{a_1+b_1+a_2+b_2+\dots} \left(S_{[411]} + S_{[33]} \right) + S_{[222]} + \\
&+ \text{tr}_{2 \times 2} \left\{ \hat{\mathcal{R}}_{[51]}^{a_1} \hat{U}_{[51]} \hat{\mathcal{R}}_{[51]}^{b_1} \hat{U}_{[51]}^\dagger \hat{\mathcal{R}}_{[51]}^{a_2} \hat{U}_{[51]} \hat{\mathcal{R}}_{[51]}^{b_1} \hat{U}_{[51]}^\dagger \dots \right\} S_{[51]} + \\
&+ \text{tr}_{2 \times 2} \left\{ \hat{\mathcal{R}}_{[321]}^{a_1} \hat{U}_{[321]} \hat{\mathcal{R}}_{[321]}^{b_1} \hat{U}_{[321]}^\dagger \hat{\mathcal{R}}_{[321]}^{a_2} \hat{U}_{[321]} \hat{\mathcal{R}}_{[321]}^{b_1} \hat{U}_{[321]}^\dagger \dots \right\} S_{[321]} + \\
&+ \text{tr}_{3 \times 3} \left\{ \hat{\mathcal{R}}_{[42]}^{a_1} \hat{U}_{[42]} \hat{\mathcal{R}}_{[42]}^{b_1} \hat{U}_{[42]}^\dagger \hat{\mathcal{R}}_{[42]}^{a_2} \hat{U}_{[42]} \hat{\mathcal{R}}_{[42]}^{b_1} \hat{U}_{[42]}^\dagger \dots \right\} S_{[42]}
\end{aligned} \tag{24}$$

Here

$$\hat{\mathcal{R}}_{[6]} = q^{\varkappa_{[4]}} = q^6, \quad \hat{\mathcal{R}}_{[411]} = \hat{\mathcal{R}}_{[33]} = -q^{\varkappa_{[31]}} = -q^2, \quad \hat{\mathcal{R}}_{[222]} = q^{\varkappa_{[22]}} = 1,$$

$$\begin{aligned}
\hat{\mathcal{R}}_{[51]} &= \begin{pmatrix} q^{\varkappa_{[4]}} & 0 \\ 0 & -q^{\varkappa_{[31]}} \end{pmatrix} = \begin{pmatrix} q^6 & 0 \\ 0 & -q^2 \end{pmatrix}, \\
\hat{\mathcal{R}}_{[321]} &= \begin{pmatrix} -q^{\varkappa_{[31]}} & 0 \\ 0 & -q^{\varkappa_{[22]}} \end{pmatrix} = \begin{pmatrix} -q^2 & 0 \\ 0 & 1 \end{pmatrix}, \\
\hat{\mathcal{R}}_{[42]} &= \begin{pmatrix} q^{\varkappa_{[4]}} & 0 & 0 \\ 0 & -q^{\varkappa_{[31]}} & 0 \\ 0 & 0 & q^{\varkappa_{[22]}} \end{pmatrix} = \begin{pmatrix} q^6 & 0 & 0 \\ 0 & -q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{25}$$

and the mixing matrices \hat{U}_Q need to be calculated.

Two of them, those for the double-line diagrams [51] and [42], can be evaluated with the help of representation theory of $SU_q(2)$, but in the [321] sector at least $SU_q(3)$ would be needed. Instead of performing this calculation directly, *a la* [1, 2], we use a shortcut: determine the five parameters (angles) of the three orthogonal matrices $\hat{U}_{[51]}$, $\hat{U}_{[321]}$ and $\hat{U}_{[42]}$ by comparison with the answers for torus and composite knots and links.

3.2 Torus knots

For torus knots $T[3, n]$ with $a_1 = b_1 = \dots = a_n = b_n = 1$ one has an alternative decomposition [19]:

$$\underline{H}_{[2]}^{3,n} = \sum_{Q \vdash 6} q^{\frac{2n}{3} \varkappa_Q} C_{[2]}^Q S_Q \tag{26}$$

where the coefficients are defined from the Adams rule

$$\begin{aligned}
\widehat{Ad}_3 S_{[2]} &= \frac{p_6 + p_3^2}{2} = \sum_{Q \vdash 6} C_{[2]}^Q S_Q = S_{[6]} - S_{[5,1]} + \underline{0 \cdot S_{[42]}} + S_{[411]} + S_{[33]} - S_{[321]} + S_{[222]}, \\
\left(\widehat{Ad}_1 S_{[2]} \right)^3 &= \frac{(p_2 + p_1^2)^3}{8} = \sum_{Q \vdash 6} C_{[2]}^Q S_Q = S_{[6]} + 2S_{[5,1]} + \underline{3 \cdot S_{[42]}} + S_{[411]} + S_{[33]} + 2S_{[321]} + S_{[222]}
\end{aligned} \tag{27}$$

for knots and links, i.e. for $n = 1, 2 \pmod{3}$ and $n = 0 \pmod{3}$ respectively.

Thus for the knots, $n = 1, 2 \pmod{3}$

$$\begin{aligned}
\underline{H}_{[2]}^{3,n} &= q^{10n} S_{[6]} - q^{6n} S_{[5,1]} + \underline{0 \cdot q^{10n/3} S_{[42]}} + q^{2n} S_{[411]} + q^{2n} S_{[33]} - S_{[321]} + q^{-2n} S_{[222]} = \\
&= q^{-2n} \left(q^{12n} S_{[6]} - q^{8n} S_{[5,1]} + \underline{0 \cdot q^{16n/3} S_{[42]}} + q^{4n} S_{[411]} + q^{4n} S_{[33]} - q^{2n} S_{[321]} + S_{[222]} \right)
\end{aligned} \tag{28}$$

Note that the only would be contribution with non-integer value of $\frac{1}{3} \varkappa_Q$ (underlined) does not contribute in the case of torus knots: the Adams coefficient $C_{[2]}^{[42]} = 0$.

Looking at the coefficients in front of the fully known "singlet" terms $S_{[6]}$, $S_{[411]}$, $S_{[33]}$, $S_{[222]}$, which do not involve yet unknown mixing matrices, we see that eq.(26) differs from the correct expression by a factor of

$$\underline{H}_{[2]}^{3,n} = q^{-2n} H_{[2]}^{3,n} \tag{29}$$

For the generic single-line (symmetric) representations $[p]$ and arbitrary number m of strands one gets, comparing the coefficients in front of $S_{[pm]}$:

$$\underline{H}_{[p]}^{m,n} = q^{\frac{2n}{m}\varkappa_{[mp]} - n(m-1)\varkappa_{[2p]}} H_{[p]}^{m,n} = q^{-n(m-2)p(p-1)} H_{[p]}^{m,n} \quad (30)$$

so that there is no discrepancy for either the first fundamental representation $p = 1$ or for the case of $m = 2$ strands, when all the knots are torus.

For the links, $n = 0 \pmod{3}$

$$\begin{aligned} \underline{H}_{[2]}^{3,n} &= q^{10n} S_{[6]} + 2q^{6n} S_{[5,1]} + \underline{3 \cdot q^{10n/3} S_{[42]}} + q^{2n} S_{[411]} + q^{2n} S_{[33]} + 2S_{[321]} + q^{-2n} S_{[222]} = \\ &= q^{-2n} \left(q^{12n} S_{[6]} + 2q^{8n} S_{[5,1]} + \underline{3 \cdot q^{16n/3} S_{[42]}} + q^{4n} S_{[411]} + q^{4n} S_{[33]} + 2q^{2n} S_{[321]} + S_{[222]} \right) \end{aligned} \quad (31)$$

This time the underlined terms are non-vanishing, but since for the links $n \equiv 3$, the power is integer in this case.

Note that the coefficients are the same for knots and links in front of the terms $S_{[6]}$, $S_{[411]}$, $S_{[33]}$ and $S_{[222]}$, in full accordance with (24), because for the torus knots and links $a_i + b_i$ is either 2 or 0, i.e. always even, so that the corresponding signs ϵ_Q can not affect the answers in the torus case (however, they affect the answers for the composite knots, see s.3.6 below).

These formulas generalize those for the fundamental representation:

$$\begin{aligned} H_{[1]}^{3,n} &= q^{2n} S_{[3]} - S_{[21]} + q^{-2n} S_{[111]}, \quad n = 1, 2 \pmod{3} \\ H_{[1]}^{3,n} &= q^{2n} S_{[3]} + 2S_{[21]} + q^{-2n} S_{[111]}, \quad n = 0 \pmod{3} \end{aligned} \quad (32)$$

considered in [2].

3.3 2×2 matrices $\hat{U}_{[51]}$ and $\hat{U}_{[321]}$ from the torus knots

When mixing matrix is of the size 2×2 , it can be parameterized by a single parameter s , sine of the mixing angle, cosine c being related through $c^2 + s^2 = 1$. Then we have for an elementary building block

$$\begin{aligned} \hat{\mathcal{R}}^a \hat{U} \hat{\mathcal{R}}^b \hat{U}^\dagger &= \begin{pmatrix} \epsilon q^\varkappa & 0 \\ 0 & \tilde{\epsilon} q^{\tilde{\varkappa}} \end{pmatrix}^a \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \epsilon q^\varkappa & 0 \\ 0 & \tilde{\epsilon} q^{\tilde{\varkappa}} \end{pmatrix}^b \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \\ &= \begin{pmatrix} \epsilon^{a+b} q^{\varkappa(a+b)} c^2 + \epsilon^a \tilde{\epsilon}^b q^{\varkappa a + \tilde{\varkappa} b} s^2 & (-\epsilon^{a+b} q^{\varkappa(a+b)} + \epsilon^a \tilde{\epsilon}^b q^{\varkappa a + \tilde{\varkappa} b}) cs \\ (\tilde{\epsilon}^{a+b} q^{\tilde{\varkappa}(a+b)} - \epsilon^b \tilde{\epsilon}^a q^{\varkappa b + \tilde{\varkappa} a}) cs & \tilde{\epsilon}^{a+b} q^{\tilde{\varkappa}(a+b)} c^2 + \epsilon^b \tilde{\epsilon}^a q^{\varkappa b + \tilde{\varkappa} a} s^2 \end{pmatrix} \end{aligned} \quad (33)$$

In the case of torus knots and links $a = b = 1$ and this reduces to

$$\hat{\mathcal{R}} \hat{U} \hat{\mathcal{R}} \hat{U}^\dagger = \begin{pmatrix} q^{2\varkappa} c^2 + (\epsilon \tilde{\epsilon}) q^{\varkappa + \tilde{\varkappa}} s^2 & (-q^{2\varkappa} + (\epsilon \tilde{\epsilon}) q^{\varkappa + \tilde{\varkappa}}) cs \\ (q^{2\tilde{\varkappa}} - (\epsilon \tilde{\epsilon}) q^{\varkappa + \tilde{\varkappa}}) cs & q^{2\tilde{\varkappa}} c^2 + (\epsilon \tilde{\epsilon}) q^{\varkappa + \tilde{\varkappa}} s^2 \end{pmatrix} \quad (34)$$

and

$$\text{Tr}_{2 \times 2} \hat{\mathcal{R}} \hat{U} \hat{\mathcal{R}} \hat{U}^\dagger = (q^{2\varkappa} + q^{2\tilde{\varkappa}}) c^2 + 2(\epsilon \tilde{\epsilon}) q^{\varkappa + \tilde{\varkappa}} s^2 = (q^{2\varkappa} + q^{2\tilde{\varkappa}}) - (q^\varkappa - (\epsilon \tilde{\epsilon}) q^{\tilde{\varkappa}})^2 s^2 \quad (35)$$

Now it remains to substitute the relevant values of \varkappa , $\tilde{\varkappa}$, ϵ and $\tilde{\epsilon}$, and compare this trace with the relevant coefficient of the HOMFLY polynomial for the torus knot $T[3, 1]$ (it is essentially the unknot but realized by a non-simplest braid; since we do not need to restrict ourselves to the topological locus here, this expression is not the same as $S_{[R]}$). After that one can calculate the traces of powers of this matrix and check that with the same value of s they reproduce the values of the coefficient for all other torus knots and links $T[3, n]$ with different n . This is, in fact, not a problem, because one should just check that with the right value of s the matrix $\hat{\mathcal{R}} \hat{U} \hat{\mathcal{R}} \hat{U}^\dagger$ has appropriate eigenvalues, proportional to the roots of unity. Finally, the same value of s determines the coefficient for all other 3-strand braids $(a_1, b_1, a_2, b_2, \dots)$.

The case of $R = [1]$ and the term $S_{[21]}$. We start with this already known case, [2] for illustrative purposes. One has to substitute $\varkappa = \varkappa_{[2]} = 1$, $\tilde{\varkappa}_{[11]} = -1$, $\epsilon = 1$, $\tilde{\epsilon} = -1$ and compare (35) with the value of the coefficient in front of S_{21} in (32) with $n = 1$, which is -1 . This gives:

$$q^2 + q^{-2} - (q + q^{-1})^2 s^2 = -1 \implies s = \frac{\sqrt{q^2 + 1 + q^{-2}}}{q + q^{-1}} = \frac{\sqrt{[3]}}{[2]} = s_2, \quad c = \frac{1}{q + q^{-1}} = \frac{1}{[2]} = c_2 \quad (36)$$

This reproduces the answer (7) for U_2 from [2].

It is easy to check that, with this values of s and c ,

$$\det_{2 \times 2}(\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger - \lambda I) = \frac{\lambda^3 - 1}{\lambda - 1} = \lambda^2 + \lambda + 1 \quad (37)$$

i.e. the two eigenvalues of $\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger$ are $e^{\pm \frac{2\pi i}{3}}$, so that

$$\text{Tr}_{2 \times 2}(\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger)^n = \begin{cases} -1 & \text{for } n = 1, 2 \pmod{3} \\ +2 & \text{for } n = 0 \pmod{3} \end{cases} \quad (38)$$

in full agreement with (32).

The case of $R = [2]$ and the term $S_{[51]}$. One has to substitute $\varkappa = \varkappa_{[4]} = 6$, $\tilde{\varkappa}_{[31]} = 2$, $\epsilon = 1$, $\tilde{\epsilon} = -1$ and compare (35) with the value of the coefficient in front of $S_{[51]}$ in (28) with $n = 1$, which is $-q^8$. This gives:

$$q^{12} + q^4 - (q^6 + q^2)^2 s^2 = -q^8 \implies s = \frac{\sqrt{q^4 + 1 + q^{-4}}}{q^2 + q^{-2}} = \frac{\sqrt{[3]_{q^2}}}{[2]_{q^2}}, \quad c = \frac{1}{q^2 + q^{-2}} = \frac{1}{[2]_{q^2}} \quad (39)$$

where $[x]_{q^2} \equiv \frac{q^{2x} - q^{-2x}}{q^2 - q^{-2}}$.

Again, it is a simple exercise to check that with these values of s and c

$$\det_{2 \times 2}(\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger - \lambda I) = \lambda^2 + q^8 \lambda + q^{16} \quad (40)$$

i.e. the two eigenvalues of $\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger$ are $q^8 e^{\pm \frac{2\pi i}{3}}$ and

$$\text{Tr}_{2 \times 2}(\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger)^n = \begin{cases} -q^{8n} & \text{for } n = 1, 2 \pmod{3} \\ +2q^{8n} & \text{for } n = 0 \pmod{3} \end{cases} \quad (41)$$

in full agreement with (28) and (31).

The case of $R = [2]$ and the term $S_{[321]}$. One has to substitute $\varkappa = \varkappa_{[31]} = 2$, $\tilde{\varkappa}_{[22]} = 0$, $\epsilon = -1$, $\tilde{\epsilon} = 1$ and compare (35) with the value of the coefficient in front of $S_{[321]}$ in (28) with $n = 1$, which is $-q^2$. This gives:

$$q^4 + 1 - (q^2 + 1)^2 s^2 = -q^2 \implies s = \frac{\sqrt{q^2 + 1 + q^{-2}}}{q + q^{-1}} = \frac{\sqrt{[3]}}{[2]}, \quad c = \frac{1}{q + q^{-1}} = \frac{1}{[2]} \quad (42)$$

With these values of s and c

$$\det_{2 \times 2}(\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger - \lambda I) = \lambda^2 + q^2 \lambda + q^4 \quad (43)$$

i.e. the two eigenvalues of $\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger$ are $q^2 e^{\pm \frac{2\pi i}{3}}$ and

$$\text{Tr}_{2 \times 2}(\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger)^n = \begin{cases} -q^{2n} & \text{for } n = 1, 2 \pmod{3} \\ +2q^{2n} & \text{for } n = 0 \pmod{3} \end{cases} \quad (44)$$

again in excellent agreement with (28) and (31).

3.4 Constraining the 3×3 matrix $\hat{U}_{[42]}$ from the torus knots

When orthogonal mixing matrix is of the size 3×3 , it can be parameterized by three independent Euler angles, namely by their sines and cosines:

$$\hat{U} = \begin{pmatrix} c_1 & 0 & s_1 \\ 0 & 1 & 0 \\ -s_1 & 0 & c_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_3 & 0 & s_3 \\ 0 & 1 & 0 \\ -s_3 & 0 & c_3 \end{pmatrix} \quad (45)$$

One now needs to perform the same trick: to compare the traces of powers of $\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger$, where $\hat{\mathcal{R}}$ is given in (25) with the known coefficients in front of $S_{[42]}$ in (28). This comparison tells that

$$\text{tr}_{3 \times 3}(\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger)^n = \begin{cases} 0 & \text{for } n = 1, 2 \pmod{3} \\ 3q^{16n/3} & \text{for } n = 0 \pmod{3} \end{cases} \quad (46)$$

for diagonal

$$\hat{\mathcal{R}} = \begin{pmatrix} q^6 & 0 & 0 \\ 0 & -q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (47)$$

The choice of the Euler decomposition in (45) is obviously adjusted to this form of the matrix $\hat{\mathcal{R}}$. At $q = 1$ a solution is obvious:

$$\hat{\mathcal{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies \hat{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}, \quad \hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c^2 - s^2 & -2cs \\ 0 & 2cs & c^2 - s^2 \end{pmatrix} \quad (48)$$

i.e. one gets the rotation matrix with the doubled angle ϕ , $s = \sin \phi$. Then (46) means that $6\phi = 2\pi k$ with any integer k , i.e. $\phi = \frac{\pi k}{3}$, and $c = \pm \frac{1}{2}$, $s = \pm \frac{\sqrt{3}}{2}$. Of course, at $q = 1$ there is a huge degeneracy: any rotation involving only the first and the third lines leaves $\hat{\mathcal{R}}(q = 1)$ intact, and one can take many other \hat{U} , obtained

by such a rotation, for example, $\hat{U} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with the same c and s . For $q = 1$ only one of the three

Euler angles in U is fixed by conditions (46).

At $q \neq 1$ conditions (46) imply that the three eigenvalues of $\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger$ are three cubic roots of unity times $q^{16/3}$, i.e. that

$$\det_{3 \times 3}(\hat{\mathcal{R}}\hat{U}\hat{\mathcal{R}}\hat{U}^\dagger - \lambda I) = q^{16} - \lambda^3 \quad (49)$$

Clearly, these are only two conditions, so that only two of the three Euler angles will be fixed by (46). One extra condition, not just coming from the 3-strand torus knot and link polynomials, will be needed to fix $\hat{U}_{[42]}$ unambiguously.

We impose this condition by making an "educated guess" that $c_3 = c_1$ and $s_3 = -s_1$. Then

$$c_1 = c_3 = \frac{1 + q^4}{\sqrt{(2q^4 - q^2 + 2)(1 + q^2 + q^4)}}, \quad c_2 = -\frac{q^4 - q^2 + 1}{1 + q^4} = -\frac{1 + q^6}{(1 + q^2)(1 + q^4)} \quad (50)$$

(the sign in c_2 is essential).

One can use an alternative parametrization instead of (45):

$$\hat{U} = \begin{pmatrix} c'_1 & 0 & s'_1 \\ 0 & 1 & 0 \\ -s'_1 & 0 & c'_1 \end{pmatrix} \begin{pmatrix} c'_2 & s'_2 & 0 \\ -s'_2 & c'_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c'_3 & 0 & s'_3 \\ 0 & 1 & 0 \\ -s'_3 & 0 & c'_3 \end{pmatrix} \quad (51)$$

In this case the Euler angles are given by

$$c'_1 = c'_3 = \sqrt{\frac{1 - q^{10}}{(2q^4 - q^2 + 2)(1 - q^6)}} = -s_1 = s_3, \quad c'_2 = -\frac{q^4 - q^2 + 1}{1 + q^4} = -\frac{1 + q^6}{(1 + q^2)(1 + q^4)} = c_2 \quad (52)$$

i.e. are "dual" to those for (45). In both cases one obtains the same matrix \hat{U} , see eq.(54) below.

It remains an open question, whether a nicer decomposition exists for this rather sophisticated mixing matrix.

3.5 The final answer

Substituting into (24) the values of the mixing angles, found in the previous subsections one finally obtains for arbitrary 3-strand braid:

$$\begin{aligned}
H_{[2]}^{a_1, b_1 | a_2, b_2 | \dots} &= q^{6(a_1 + b_1 + a_2 + b_2 + \dots)} S_{[6]} + (-q^2)^{a_1 + b_1 + a_2 + b_2 + \dots} (S_{[411]} + S_{[33]}) + S_{[222]} + \\
&+ \text{tr}_{2 \times 2} \left\{ \left(\begin{pmatrix} q^6 & 0 \\ 0 & -q^2 \end{pmatrix} \right)^{a_1} \begin{pmatrix} -\frac{1}{[2]_{q^2}} & \frac{\sqrt{[3]_{q^2}}}{[2]_{q^2}} \\ -\frac{\sqrt{[3]_{q^2}}}{[2]_{q^2}} & -\frac{1}{[2]_{q^2}} \end{pmatrix} \left(\begin{pmatrix} q^6 & 0 \\ 0 & -q^2 \end{pmatrix} \right)^{b_1} \begin{pmatrix} -\frac{1}{[2]_{q^2}} & -\frac{\sqrt{[3]_{q^2}}}{[2]_{q^2}} \\ \frac{\sqrt{[3]_{q^2}}}{[2]_{q^2}} & -\frac{1}{[2]_{q^2}} \end{pmatrix} \dots \right\} S_{[51]} + \\
&+ \text{tr}_{2 \times 2} \left\{ \left(\begin{pmatrix} -q^2 & 0 \\ 0 & 1 \end{pmatrix} \right)^{a_1} \begin{pmatrix} -\frac{1}{[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ -\frac{\sqrt{[3]_q}}{[2]_q} & -\frac{1}{[2]_q} \end{pmatrix} \left(\begin{pmatrix} -q^2 & 0 \\ 0 & 1 \end{pmatrix} \right)^{b_1} \begin{pmatrix} -\frac{1}{[2]_q} & -\frac{\sqrt{[3]_q}}{[2]_q} \\ \frac{\sqrt{[3]_q}}{[2]_q} & -\frac{1}{[2]_q} \end{pmatrix} \dots \right\} S_{[321]} + \\
&+ \text{tr}_{3 \times 3} \left\{ \left(\begin{pmatrix} q^6 & & \\ & -q^2 & \\ & & 1 \end{pmatrix} \right)^{a_1} U_{[42]} \left(\begin{pmatrix} q^6 & & \\ & -q^2 & \\ & & 1 \end{pmatrix} \right)^{b_1} U_{[42]}^\dagger \dots \right\} S_{[42]}
\end{aligned} \tag{53}$$

The matrix $U_{[42]}$ is equal to:

$$\begin{aligned}
&\begin{pmatrix} \frac{q^4}{(q^4+1)(q^4+q^2+1)} & -\frac{q\sqrt{q^8+q^6+q^4+q^2+1}}{(q^4+1)\sqrt{q^4+q^2+1}} & -\frac{\sqrt{q^8+q^6+q^4+q^2+1}}{q^4+q^2+1} \\ \frac{q\sqrt{q^8+q^6+q^4+q^2+1}}{(q^4+1)\sqrt{q^4+q^2+1}} & -\frac{q^4-q^2+1}{q^4+1} & \frac{q}{\sqrt{q^4+q^2+1}} \\ -\frac{\sqrt{q^8+q^6+q^4+q^2+1}}{q^4+q^2+1} & -\frac{q}{\sqrt{q^4+q^2+1}} & \frac{q^2}{q^4+q^2+1} \end{pmatrix} = \\
&= \begin{pmatrix} \frac{[2]}{[3][4]} & -\frac{[2]}{[4]}\sqrt{\frac{[5]}{[3]}} & -\frac{\sqrt{[5]}}{[3]} \\ \frac{[2]}{[4]}\sqrt{\frac{[5]}{[3]}} & -\frac{[6]}{[3][4]} & \frac{[1]}{\sqrt{[3]}} \\ -\frac{\sqrt{[5]}}{[3]} & -\frac{1}{\sqrt{[3]}} & \frac{1}{[3]} \end{pmatrix}
\end{aligned} \tag{54}$$

This is the same matrix as the matrix of the Racah coefficients, (A.20) in [22].

3.6 Composite knots and links: a check

In this section we perform further checks, making use of topological equivalence between different braids, i.e. homotopic equivalence of the corresponding knots and links. Accordingly, in this section we can consider only the ordinary HOMFLY polynomials H reduced to the topological locus

$$p_k = p_k^* = \frac{A^k - A^{-k}}{q^k - q^{-k}} \tag{55}$$

The case of $b_1 = a_2 = b_2 = \dots = 0$: In this simplest example, with only one non-vanishing parameter a_1 , the 3-strand knot/link splits into untied a 2-strand knot/link and the unknot. Accordingly,

$$H_R^{3, (a, 0, 0, \dots)} = H_R^{2, a} \cdot H_R^0 \tag{56}$$

At the same time, in this case (53) is also drastically simplified: all mixing matrices drop away from the formula and it reduces to just

$$\begin{aligned} H_{[2]}^{3,(a,0,0,0,\dots)} &= q^{6a} S_{[6]} + (-q^2)^a (S_{[411]} + S_{[33]}) + S_{[222]} + \\ &+ (q^{6a} + (-q^2)^a) S_{[51]} + ((-q^2)^a + 1) S_{[321]} + (q^{6a} + (-q^2)^a + 1) S_{[42]} \end{aligned} \quad (57)$$

Note that, in variance with expressions for the 3-strand *torus* knots and links, this formula is sensitive to the sign of the R -matrix eigenvalue $-q^2$. It remains to reduce (57) to the topological locus (55), where the Schur functions turn into the quantum dimensions, and check that this coincides with the r.h.s. of (56) with $R = [2]$, where the unknot polynomial is just $H_R^0 = S_R^*$ and $H_{[2]}^{2,a}$ is given by the second line of (13). Of course, such a relation can *not* be lifted to the entire p -space: (57) does *not* coincide with $H_{[2]}^{(2,a)} S_{[2]}$ beyond the topological locus (55): one suffices to note that the former depends on p_6 , while the latter one does not.

3.7 Results

3.7.1 The figure eight knot 4_1

This knot can be realized with the braid

$$4_1 : \quad (a_1, b_1 | a_2, b_2) = (1, -1 | 1, -1), \quad (58)$$

similar to a possible 3-strand realization of the trefoil, which is a torus knot $T[2, 3] = T[3, 2]$

$$3_1 : \quad (1, 1 | 1, 1) \quad (59)$$

In the fundamental representation one had

$$H_{[1]}^{4_1} = S_{[3]}^* + (q^4 - 2q^2 + 1 - 2q^{-2} + q^{-4}) S_{[21]}^* + S_{[111]}^* = (A^2 - (q^2 - 1 + q^{-2}) + A^{-2}) S_{[1]}^* \quad (60)$$

while

$$H_{[1]}^{3_1} = q^4 S_{[3]}^* - S_{[21]}^* + q^{-4} S_{[111]}^* = ((q^2 + q^{-2}) A^2 - 1) S_{[1]}^* \quad (61)$$

The second expression is highly asymmetric, while the formula for 4_1 is *very* symmetric even when expressed in terms of the A variable: this is a specifics of 4_1 .

In the symmetric representation $R = [2]$ the answer is

$$\begin{aligned} H_{[2]}^{4_1} &= (q^4 A^4 - (1 + q^2)(1 - q^2 + q^6) q^{-2} A^2 + (q^6 - q^4 + 3 - q^{-4} + q^{-6}) - \\ &- (1 + q^{-2})(1 - q^{-2} + q^{-6}) q^2 A^{-2} + q^{-4} A^{-4}) S_{[2]}^* \end{aligned} \quad (62)$$

This can be compared with the asymmetric formula for the trefoil $(1, 1, 1, 1)$

$$H_{[2]}^{3_1} = q^8 (A^4(1 + q^4 + q^6 + q^{12}) - A^2(1 + q^2)(1 + q^6) + q^2) S_{[2]}^* \quad (63)$$

Expression (62) certainly coincide with results presented in existing literature, see e.g. [8]. Moreover, it turns out that in the case of 4_1 one can get the result for *any* symmetric $[p]$ and antisymmetric $[1^p]$ representation [3].

The HOMFLY polynomials in the symmetric representation for other 3-strand knots with no more than 8 crossings are collected in the Appendix.

3.8 Antisymmetric representation

In order to construct the HOMFLY polynomials in the antisymmetric representation $[1, 1]$, one could repeat the standard machinery of the mixing matrices etc we described above. However, the result can be obtained much simpler using a symmetry of the HOMFLY polynomials.

Indeed, the character expansions of the HOMFLY polynomials possess a Z_2 -symmetry

$$A, q, S_R^* \longleftrightarrow A, -\frac{1}{q}, S_{R'}^* \quad (64)$$

where R' is a transposition of the Young diagram R . This symmetry can be easily understood, since $S_{R'}\{p_k\} = S_R\{(-)^{k-1} p_k\}$ and $\kappa_R = -\kappa_{R'}$. At the same time, all the $(SU_q(N))$ group representation quantities (in particular, the mixing matrices) are also possess this antipodal symmetry. Hence, one can calculate the HOMFLY polynomials in the antisymmetric representation just making a substitution $q \rightarrow -1/q$ in the HOMFLY polynomials for the symmetric representation obtained in the previous sections.

3.9 Ooguri-Vafa conjecture

In the paper [23], the authors conjectured a connection of the Chern-Simons theory with topological string on the resolution of the conifold. In fact, they proposed that the generating function Z of average of the Wilson loop in different representations is associated with the topological string partition function Z_{str} . In accordance with the Ooguri-Vafa result [24] Z is given by the sum³

$$Z(q, A, K) = \sum_R \chi_R(p) H_R^K(q, A) \quad (65)$$

where the sum runs over all the irreducible representation of $SU(N)$ ($A = q^N$). Now the topological nature of this object implies that the "connected" correlators $f_R(q, A)$ defined by the expansion

$$\log Z = \sum_{n=0, R} \frac{1}{n} f_R(q^n, A^n) \chi_R(p^{(n)}) \quad (66)$$

where the set of variables $p_k^{(n)} \equiv p_{nk}$, has the generic structure

$$f_R(q, A) = \sum_{n, k} \tilde{N}_{R, n, k} \frac{A^n q^k}{q - q^{-1}} \quad (67)$$

$\tilde{N}_{R, n, k}$ are integer and the parity of n in the sum coincides with the parity of $|R|$ while the parity of k is inverse. These numbers are related to the Gopakumar-Vafa integers $n_{\Delta, n, k}$ [24] by the relation

$$n_{\Delta, n, k} = \sum_R \Phi_R(\Delta) \tilde{N}_{R, n, k}$$

where $\Phi_R(\Delta)$ is the character of the symmetric group $S_{|\Delta|}$. The integers $\tilde{N}_{R, n, k}$ are more refined, since their integrality implies that $n_{\Delta, n, k}$ are integer but not vice verse. In fact, one can consider even more refined integers [25]

$$f_R(q, A) = \sum_{n, k, R_1, R_2} C_{RR_1 R_2} \Sigma_{R_1}(q) N_{R_2, n, k} A^n (q^{-1} - q)^{2k-1} \quad (68)$$

where

$$C_{RR_1 R_2} = \sum_{\Delta} \frac{\Phi_R(\Delta) \Phi_{R_1}(\Delta) \Phi_{R_2}(\Delta)}{z_{\Delta}} \quad (69)$$

the Clebsh-Gordon coefficients of the symmetric group, z_{Δ} is the standard symmetric factor of the Young diagram [5] and $\Sigma_R(q)$ is a monomial non-zero only for the corner Young diagrams $R = [l - d, 1^d]$ and is equal to

$$\Sigma_R(q) = (-1)^d q^{2d-l+1} \quad (70)$$

First few terms for f_R and $N_{R, n, k}$ are

$$\begin{aligned} f_{[1]}(q, A) &= H_{[1]}(q, A) \\ f_{[2]}(q, A) &= H_{[2]}(q, A) - \frac{1}{2} (H_{[1]}(q, A)^2 + H_{[2]}(q^2, A^2)) \\ f_{[1,1]}(q, A) &= H_{[1,1]}(q, A) - \frac{1}{2} (H_{[1]}(q, A)^2 - H_{[2]}(q^2, A^2)) \\ &\dots \end{aligned} \quad (71)$$

and

$$\begin{aligned} f_{[1]}(q, A) &= \sum_{n, k} N_{[1], n, k} (q^{-1} - q)^{2k-1} A^n \\ f_{[2]}(q, A) &= \sum_{n, k} (q^{-1} N_{[2], n, k} - q N_{[1,1], n, k}) (q^{-1} - q)^{2k-1} A^n \\ f_{[1,1]}(q, A) &= \sum_{n, k} (-q N_{[2], n, k} + q^{-1} N_{[1,1], n, k}) (q^{-1} - q)^{2k-1} A^n \\ &\dots \end{aligned} \quad (72)$$

³ This sum can be obtained as the Chern-Simons average of the Ooguri-Vafa operator $\exp \sum_n \frac{1}{n} \text{Tr} (f_K A dx)^n \text{Tr} V^n$, where A is the gauge field, $p_k = \text{Tr} V^k$ are external sources and the traces are taken over the fundamental representation.

We calculate both the Ooguri-Vafa polynomials $f_{[2]}(q, A)$ and the numbers $N_{[2],n,k}$ for all 3-strand knots with no more than 8 crossings in the Appendix. The integrality of these numbers and their using in product formulas is discussed in [26].

3.10 "Special" polynomials

The "special" polynomials are defined [20] as the limit of ratio of the HOMFLY polynomials and the quantum dimensions as $q \rightarrow 1$:

$$\mathfrak{H}_R^\mathcal{K}(A) = \lim_{q \rightarrow 1} \frac{H_R^\mathcal{K}(q, A)}{S_R^*(q, A)} \quad (73)$$

Note that the limit is taken with fixed A , and both the HOMFLY polynomial H_R and the quantum dimension S_R^* are singular behaving as $(q - q^{-1})^{-|R|}$. Here $|R|$ is the number of boxes in the Young diagram R . Note that in this limit

$$\lim_{q \rightarrow 1} S_R(A) = d_R S_{[1]}(A)^{|R|} \quad (74)$$

where

$$d_R = S_R\{p\} \Big|_{p_k = \delta_{k,1}} = \prod_{(i,j) \in R} \frac{1}{h_{i,j}} \quad (75)$$

and $h_{i,j}$ is the "hook" length.

The conjectured property of the "special" polynomials reads as [20, 27]

$$\mathfrak{H}_R^\mathcal{K}(A) = \left(\mathfrak{H}_{[1]}^\mathcal{K}(A) \right)^{|R|} \quad (76)$$

and is presumably valid for arbitrary \mathcal{K} and R .

For example,

$$\mathfrak{H}_{[2]}^{3_1}(A) = (2A^2 - 1)^2, \quad (77)$$

$$\mathfrak{H}_{[1]}^{3_1}(A) = 2A^2 - 1,$$

$$\mathfrak{H}_{[2]}^{4_1}(A) = \left(A^2 - 1 + A^{-2} \right)^2, \quad (78)$$

$$\mathfrak{H}_{[1]}^{4_1}(A) = A^2 - 1 + A^{-2}$$

etc.

This conjecture is an amusing "dual" of a somewhat similar conjecture

$$\mathfrak{A}_R^\mathcal{K}(q) = \mathfrak{A}_{[1]}^\mathcal{K} \left(q^{|R|} \right) \quad (79)$$

for the Alexander polynomial

$$\mathfrak{A}_R^\mathcal{K}(q) = \lim_{A \rightarrow 1} \frac{H_R^\mathcal{K}(q, A)}{S_R^*} \quad (80)$$

We check these two conjectures for the concrete knots in the Appendix.

Similarly, one can consider the "special" limit of $q \rightarrow 1$ for other polynomials, e.g. for the Ooguri-Vafa polynomials $f_R(q, A)$. The special Ooguri-Vafa polynomials $\mathfrak{f}_R(A) \equiv \lim_{q \rightarrow 1} \frac{f_R(q, A)}{S_{[R]}^*}$, however, depend on the representation much less trivially than the "special" and Alexander polynomials (see the Appendix for examples). Note that $\mathfrak{f}_{[2]}(A) = -\mathfrak{f}_{[11]}(A)$.

3.11 Framing factor

In this text we assume that the \mathcal{R} -matrix is normalized so that in the channel $Q \in R \otimes R$ its eigenvalue is equal to $\pm q^{\pm Q}$ and is independent of R . This simplifies our formulas, and this is important for their extension beyond the topological locus (2). However, instead this breaks some properties, important for the knot theory, including topological (ambient isotopy) invariance. Still, this difference is very easy to take into account by adding an overall factor, which is simple, but depends on representation and even on the rank of the group $SU(N)$. This factor is also important in the definition of the Ooguri-Vafa polynomials⁴ and is ambiguously determined due to

⁴For the fully symmetric knots like the figure eight 4_1 with the vanishing writhe number the factor is unity and the Ooguri-Vafa polynomials can be easily extended beyond the topological locus [3]. For generic knots this extension needs a separate discussion.

the freedom in choosing the framing [28]. We choose the standard, or canonical framing. Then, the \mathcal{R} -matrix, which is adequate for knot theory calculations is actually normalized differently:

$$\mathcal{R}_{R \otimes R}^{norm} = A^{-|R|} q^{-4\kappa_R} \mathcal{R}_{R \otimes R} \quad (81)$$

This means that all our answers for the HOMFLY polynomials should be multiplied by the additional factor

$$H_R^{\mathcal{B}} \longrightarrow H_R^{\mathcal{K}} = \left(A^{|R|} q^{4\kappa_R} \right)^{-w^{\mathcal{B}}} H_R^{\mathcal{B}} \{p_k^*\} \quad (82)$$

where $w^{\mathcal{B}}$ is the algebraic number of intersections in the braid \mathcal{B} called *the writhe number*. We illustrate the significance of this factor by three examples. First of them concerns the topological invariance, the second one the identity (79) for the Alexander polynomials: in both cases the additional factor is essential. The third example demonstrates that (82) is consistent with existing literature.

Example 1: If the torus knot 3_1 is represented by the 2-strand braid $[2, 3] = (1)^3 = (1, 1, 1)$, then one gets for the HOMFLY polynomial

$$\frac{q^3 {}^*S_{[2]} - q^{-3} {}^*S_{[11]}}{{}^*S_{[1]}} = \frac{q^3(Aq - A^{-1}q^{-1}) - q^{-3}(Aq^{-1} - A^{-1}q)}{q^2 - q^{-2}} = A(q^2 + q^{-2}) - A^{-1} \quad (83)$$

If the same knot is represented by the 3-strand braid $[3, 2] = (1, 1)^2 = (1, 1|1, 1)$, one gets instead

$$\frac{q^4 {}^*S_{[3]} - \text{tr}_{2 \times 2} \left(\hat{\mathcal{R}} \hat{\mathcal{U}} \hat{\mathcal{R}} \hat{\mathcal{U}}^\dagger \right)^2 {}^*S_{[21]} + q^{-4} {}^*S_{[111]}}{{}^*S_{[1]}} = A^2(q^2 + q^{-2}) - 1 \quad (84)$$

Clearly, these two expressions do not coincide and differ by a factor of A , which is exactly taken into account by the correction factor (82), because $w^{[3,2]} = 4$, while $w^{[2,3]} = 3$, and in this example $\kappa_{[1]} = 0$. For $p = 2$ the two HOMFLY polynomials, calculated in this paper, differ by a factor of $A^2 q^4$, which is again nicely eliminated by (82), because $\kappa_{[2]} = 1$. Note that in the Appendix we choose the opposite orientation for the trefoil: $(-1, -1, -1)$ in order to better match formulas from the standard knot tables.

Example 2: In fact, the Alexander polynomials made from our extended HOMFLY polynomials,

$$\mathfrak{A}_{[p]}^{[2,n]}(q) = \lim_{A \rightarrow 1} \frac{q^{p(2p-1)n} {}^*S_{[2p]} - q^{p(2p-3)n} {}^*S_{[2p-1,1]}}{{}^*S_{[p]}} \quad (85)$$

(all other Young diagrams from the decomposition of $[p] \otimes [p]$ do not contribute at $A = 1$), satisfy

$$\mathfrak{A}_{[p]}^{[2,n]}(q) = q^{2p(p-1)n} \mathfrak{A}_{[1]}^{[2,n]}(q^p) = q^{4\kappa_{[p]} \cdot w^{[2,n]}} \mathfrak{A}_{[1]}^{[2,n]}(q^p) \quad (86)$$

rather than (79). Unwanted factors in this relation are eliminated after the factor (82) is taken into account.

Example 3: As we know from (30), the torus polynomial character expansion of [19] based on use of the Adams operation, and also suitable for continuation from the topological locus (2) to the entire space of time-variables, differs from ours by a factor of $q^{-n(m-2)p(p-1)}$. The normalized HOMFLY obtained from ours by the rule (82) should, therefore, differ from that one by a factor of $q^{-2n(m-1)p(p-1)} \cdot q^{n(m-2)p(p-1)} = q^{-nmp(p-1)}$ (since $w^{[m,n]} = (m-1)n$ and $\kappa_{[p]} = \frac{p(p-1)}{2}$):

$$H_{[p]}^{[m,n]} = A^{-(m-1)np} q^{-mnp(p-1)} \underline{H}_{[p]}^{m,n} = A^{-(m-1)n|R|} q^{-2mn\kappa_{[p]}} \underline{H}_{[p]}^{m,n} \quad (87)$$

This is exactly the factor used in [19] for arbitrary representation R . It can deserve noting that mn is *not* the writhe number of the braid associated with the torus knot, and coefficient 2 is different from 4 in (82).

3.12 Cabling

The standard way to obtain the colored HOMFLY polynomials is to extract them from those in the fundamental representation, but for different knots and links. Namely, if one needs $\mathcal{H}_R^{\mathcal{K}}$, one considers instead $\mathcal{H}_{[1]}^{\mathcal{K}^{[R]}}$, where $\mathcal{K}^{[R]}$ is the *cabling* of the knot \mathcal{K} , obtained by substituting the knot with a set of $|R|$ parallel ones (a "cable"),

i.e. actually a knot \mathcal{K} is substituted by an $|R|$ -component link. However, to extract information about an arbitrary R of a given size $|R|$ one should also allow additional intertwinings of the wires inside each cable, which decreases the number of components in the link, so that $\mathcal{K}^{|R|}$ is actually a linear combination of several links, made in this way from the $|R|$ -cabled \mathcal{K} .

If the knot \mathcal{K} is represented by an m -strand braid, the cabling involves $m|R|$ strands. Since general formulas are known [2] for arbitrary r -strand knots in the fundamental representations, one can actually demonstrate how the cabling procedure works for arbitrary 2-strand knots in symmetric and antisymmetric representations. For the 3-strand knots in these representations or for 2-strand knots in representations [3], [21], [111] one needs the knowledge of the 6-strand knots in the fundamental representation, which is still not available in full generality. Thus, in the rest of this section we rederive $\mathcal{H}_{[2]}^{[2,n]}$ and $\mathcal{H}_{[11]}^{[2,n]}$ from $\mathcal{H}_{[1]}^{[2,n]^2}$.

Cabling is a tedious, but well known and widely used procedure, we add this subsection for the sake of completeness.

Cabling the unknot

Our first example is actually the 1-strand knot: the unknot. The 2-cabling of a 1-strand braid implies that it is substituted with a 2-strand one: the two unlinked unknots and the answer is

$$H_{[1]}^{[2,0]}\{p_k\} = S_{[1]}^2\{p_k\} = S_{[2]}\{p_k\} + S_{[11]}\{p_k\} = H_{[2]}^{[1,0]}\{p_k\} + H_{[11]}^{[1,0]}\{p_k\} \quad (88)$$

a linear combination of unknot polynomials in two representations of the size $|R| = 2$. Similarly, the untwisted p -cabling gives a linear combination

$$H_{[1]}^{[p,0]}\{p_k\} = S_{[1]}^p\{p_k\} = \sum_{R: |R|=p} \mathcal{H}_R^{[1,0]}\{p_k\} \quad (89)$$

To extract the individual HOMFLY polynomials for two representations [2] and [11] one needs to consider not only the two non-intersecting strands, but also to allow one intertwining. One would naturally assume that one extra intersection just provides $S_{[2]} - S_{[11]}$, but this is the case only for $q = 1$. If one associates an extra R -matrix with this additional intersection, one gets q -dependent factors:

$$H_{[1]}^{[2,1]}\{p_k\} \stackrel{(13)}{=} qS_{[2]}\{p_k\} - q^{-1}S_{[11]}\{p_k\} = qH_{[2]}^{[1,0]}\{p_k\} - q^{-1}H_{[11]}^{[1,0]}\{p_k\} \quad (90)$$

and finally the cabling of the unknot implies

$$\begin{aligned} H_{[2]}^{[1,0]}\{p_k\} &= \frac{1}{1+q^2}H_{[1]}^{[2,0]}\{p_k\} + \frac{q}{1+q^2}H_{[1]}^{[2,1]}\{p_k\}, \\ H_{[11]}^{[1,0]}\{p_k\} &= \frac{q^2}{1+q^2}H_{[1]}^{[2,0]}\{p_k\} - \frac{q}{1+q^2}H_{[1]}^{[2,1]}\{p_k\} \end{aligned} \quad (91)$$

If one restricts the answer to the topological locus *and restore the factors*, see s.3.11, to make contact with the standard calculations, one would write the same relations as follows:

$$\begin{aligned} H_{[2]}^{[1,0]}(A|q) &= \frac{1}{1+q^2}H_{[1]}^{[2,0]}(A|q) + \frac{qA}{1+q^2}\left(A^{-1}H_{[1]}^{[2,1]}(A|q)\right), \\ H_{[11]}^{[1,0]}(A|q) &= \frac{q^2}{1+q^2}H_{[1]}^{[2,0]}(A|q) - \frac{qA}{1+q^2}\left(A^{-1}H_{[1]}^{[2,1]}(A|q)\right) \end{aligned} \quad (92)$$

Formulas (91) and (92) actually define *the projectors*, specifying the linear combinations of cabled knots with additional twistings [19], which select particular representations [2] and [11]. Since they are actually independent of the knot, the same projectors are used for the same purpose below, when we switch to a little more interesting examples of 2-cabling the 2-strand knots.

2-cabling the 2-strand knots

A new thing as compared to the previous subsection is that one has intersections in original 2-strand braid (there were none in the 1-strand case). Each \mathcal{R} -matrix at the 2-strand crossing is substituted by four \mathcal{R} -matrices, after lifting to 4 strands:

$$\mathcal{R} \longrightarrow (\mathcal{R} \otimes I \otimes I) (\mathcal{R} \otimes \mathcal{R}) (I \otimes I \otimes \mathcal{R}) \quad (93)$$

so that the 2-strand braid $[2, n]$ is lifted to a 4-strand braid of the type $(0, 1, 1|1, 1, 0)^n$. Moreover, to separate representations $[2]$ and $[11]$ one also needs to allow one twisting between the first two and the last two braids, i.e. to consider the four slightly different links/knots

$$(0, 1, 1|1, 1, 0)^n, \quad (0, 1, 1|1, 1, 0)^n(1, 0, 0), \quad (0, 1, 1|1, 1, 0)^n(0, 0, 1), \quad (0, 1, 1|1, 1, 0)^n(1, 0, 1) \quad (94)$$

Making use of projectors (91) and (92), one gets, in somewhat compressed notation:

$$\begin{aligned} q^{-2n} H_{[2]}^{[2,n]} \{p_k\} &= \frac{1}{(1+q^2)^2} \left(H_{[1]}^{[4,(000),n]} \{p_k\} + q H_{[1]}^{[4,(001),n]} \{p_k\} + q H_{[1]}^{[4,(100),n]} \{p_k\} + q^2 H_{[1]}^{[4,(101),n]} \{p_k\} \right), \\ q^{2n} H_{[11]}^{[2,n]} \{p_k\} &= \frac{1}{(1+q^2)^2} \left(q^4 H_{[1]}^{[4,(000),n]} \{p_k\} - q^3 H_{[1]}^{[4,(001),n]} \{p_k\} - q^3 H_{[1]}^{[4,(100),n]} \{p_k\} + q^2 H_{[1]}^{[4,(101),n]} \{p_k\} \right) \end{aligned} \quad (95)$$

According to [2], substituting the peculiar braid $(a_1 b_1 c_1 | a_2 b_2 c_2 | a_3 b_3 c_3 | \dots) = \underbrace{(011|110| \dots | 011|110)}_{n \text{ times}}$ into the general formula [2, eq.(65)] for the 4-strand extended HOMFLY polynomials gives

$$\begin{aligned} H_{[1]}^{[4,(000),n]} &= H_{[1]}^{[4,(011|110)^n(000)]} = q^{4n} S_{[4]} + \text{tr}_{3 \times 3} \left(\hat{\mathcal{R}}_{[31]} \hat{V}_{[31]} \hat{U}_{[31]} \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]}^\dagger \hat{V}_{[31]}^\dagger \hat{U}_{[31]}^\dagger \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]} \hat{\mathcal{R}}_{[31]} \right)^n \cdot S_{[31]} + \\ &+ \left(q \longleftrightarrow -\frac{1}{q} \right) + \text{tr}_{2 \times 2} \left(\hat{\mathcal{R}}_{[22]} \hat{U}_{[22]}^\dagger \hat{\mathcal{R}}_{[22]}^2 \hat{U}_{[22]} \hat{\mathcal{R}}_{[22]} \right)^n \cdot S_{[22]} \end{aligned} \quad (96)$$

and for the other three twisted cablings:

$$\begin{aligned} H_{[1]}^{[4,(100),n]} &= q^{4n+1} S_{[4]} + \text{tr}_{3 \times 3} \left\{ \hat{U}_{[31]} \left(\hat{\mathcal{R}}_{[31]} \hat{V}_{[31]} \hat{U}_{[31]} \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]}^\dagger \hat{V}_{[31]}^\dagger \hat{U}_{[31]}^\dagger \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]} \hat{\mathcal{R}}_{[31]} \right)^n \hat{U}_{[31]}^\dagger \hat{\mathcal{R}}_{[31]} \right\} \cdot S_{[31]} + \\ &+ \left(q \longleftrightarrow -\frac{1}{q} \right) + \text{tr}_{2 \times 2} \left\{ \left(\hat{\mathcal{R}}_{[22]} \hat{U}_{[22]}^\dagger \hat{\mathcal{R}}_{[22]}^2 \hat{U}_{[22]} \hat{\mathcal{R}}_{[22]} \right)^n \hat{U}_{[22]}^\dagger \hat{\mathcal{R}}_{[22]} \hat{U}_{[22]} \right\} \cdot S_{[22]}, \\ H_{[1]}^{[4,(001),n]} &= q^{4n+1} S_{[4]} + \text{tr}_{3 \times 3} \left\{ \left(\hat{\mathcal{R}}_{[31]} \hat{V}_{[31]} \hat{U}_{[31]} \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]}^\dagger \hat{V}_{[31]}^\dagger \hat{U}_{[31]}^\dagger \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]} \hat{\mathcal{R}}_{[31]} \right)^n \hat{V}_{[31]} \hat{U}_{[31]} \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]}^\dagger \hat{V}_{[31]}^\dagger \right\} \cdot S_{[31]} + \\ &+ \left(q \longleftrightarrow -\frac{1}{q} \right) + \text{tr}_{2 \times 2} \left\{ \left(\hat{\mathcal{R}}_{[22]} \hat{U}_{[22]}^\dagger \hat{\mathcal{R}}_{[22]}^2 \hat{U}_{[22]} \hat{\mathcal{R}}_{[22]} \right)^n \hat{U}_{[22]}^\dagger \hat{\mathcal{R}}_{[22]} \hat{U}_{[22]} \right\} \cdot S_{[22]}, \\ H_{[1]}^{[4,(101),n]} &= q^{4n+2} S_{[4]} + \text{tr}_{3 \times 3} \left\{ \left(\hat{\mathcal{R}}_{[31]} \hat{V}_{[31]} \hat{U}_{[31]} \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]}^\dagger \hat{V}_{[31]}^\dagger \hat{U}_{[31]}^\dagger \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]} \hat{\mathcal{R}}_{[31]} \right)^n \hat{U}_{[31]}^\dagger \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]} \hat{V}_{[31]} \hat{U}_{[31]} \hat{\mathcal{R}}_{[31]} \hat{U}_{[31]}^\dagger \hat{V}_{[31]}^\dagger \right\} \cdot S_{[31]} + \\ &+ \left(q \longleftrightarrow -\frac{1}{q} \right) + \text{tr}_{2 \times 2} \left\{ \left(\hat{\mathcal{R}}_{[22]} \hat{U}_{[22]}^\dagger \hat{\mathcal{R}}_{[22]}^2 \hat{U}_{[22]} \hat{\mathcal{R}}_{[22]} \right)^n \hat{U}_{[22]}^\dagger \hat{\mathcal{R}}_{[22]}^2 \hat{U}_{[22]} \right\} \cdot S_{[22]} \end{aligned} \quad (97)$$

Now let us look at the coefficient in front of $S_{[4]}$. The two linear combinations, corresponding to (95) for this coefficient give just

$$\begin{aligned} \frac{q^{4n} + 2q \cdot q^{4n+1} + q^2 \cdot q^{4n+2}}{(1+q^2)^2} &= q^{4n}, \\ \frac{q^4 \cdot q^{4n} - 2q^3 \cdot q^{4n+1} + q^2 \cdot q^{4n+2}}{(1+q^2)^2} &= 0 \end{aligned} \quad (98)$$

Similarly, for two linear combinations in front of $S_{[22]}$ one has

$$\begin{aligned} \frac{(q^{2n} + q^{-2n}) + 2q \cdot (-q^{2n-1} + q^{1-2n}) + q^2 \cdot (q^{2n-2} + q^{2-2n})}{(1+q^2)^2} &= q^{-2n}, \\ \frac{q^4 \cdot (q^{2n} + q^{-2n}) - 2q^3 \cdot (-q^{2n-1} + q^{1-2n}) + q^2 \cdot (q^{2n-2} + q^{2-2n})}{(1+q^2)^2} &= q^{2n} \end{aligned} \quad (99)$$

and for those in front of $S_{[31]}$ the intermediate expressions are different for knots and links: for n odd

$$\begin{aligned} \frac{-1 + 2q \cdot (-q) + q^2 \cdot (-q^2)}{(1+q^2)^2} &= -1 \\ \frac{q^4 \cdot (-1) - 2q^3 \cdot (-q) + q^2 \cdot (-q^2)}{(1+q^2)^2} &= 0 \end{aligned} \quad (100)$$

while for n even

$$\begin{aligned} \frac{(2q^{2n} + 1) + 2q \cdot (q^{2n+1} - q^{2n-1} + q) + q^2 \cdot (q^2 - 2q^{2n})}{(1+q^2)^2} &= 1 \\ \frac{q^4 \cdot (2q^{2n} + 1) - 2q^3 \cdot (q^{2n+1} - q^{2n-1} + q) + q^2 \cdot (q^2 - 2q^{2n})}{(1+q^2)^2} &= 0 \end{aligned} \quad (101)$$

Thus, one finally obtains

$$\begin{aligned} q^{-2n}H_{[2]}^{2,n} &= q^{4n}S_{[4]} \mp S_{[31]} + q^{-2n}S_{[22]} \\ q^{2n}H_{[11]}^{2,n} &= q^{-4n}S_{[1111]} \mp S_{[211]} + q^{2n}S_{[22]} \end{aligned} \tag{102}$$

which coincides with (13).

4 Summary

In this paper we continued our program of constructing simple matrix expressions for the colored HOMFLY polynomials of arbitrary knots/links started in [1, 2]. In practice, we always deal with braid representations of knots. Here we considered the symmetric and antisymmetric representations [2] and [1, 1] for 3-strand braids. One can construct the result inductively, using the representation group theory, however, in this paper we used an indirect way of using the known answers for the torus knot/link polynomials in order to restore all the necessary ingredients (in particular, the mixing matrices) for the generic answer. We return to using the group theory approach elsewhere [18].

Using the formulas, obtained in this paper (we listed various knot polynomials for the knots that can be described by 3-strand braids with no more than 8 crossings in the Appendix) we tested various conjectures, from the Ooguri-Vafa conjecture [23] and its generalization [25] to the conjecture of the representation dependence of the "special" polynomials. The HOMFLY polynomials calculated in the paper were partly obtained earlier in a series of papers by the Indian group [8, 7] within a different though close approach. In these cases our results confirm these earlier calculations.

The results presented here are substantially extended in [18] to include higher symmetric representations but this requires a deeper insight into the structure of the mixing matrices and, hence, is beyond the scope of the present paper.

Note added

After this paper was published there appeared a paper [30] with calculations of the HOMFLY polynomials in the first symmetric representation and of the corresponding Ooguri-Vafa polynomials for various knots and links. Their results for the 3-strand knots coincide with formulas of this paper for the only exception of the HOMFLY polynomial for knot 7_5 where we made a misprint (the Ooguri-Vafa polynomial was written in our paper correctly). We are grateful to the authors of [30] for the correction.

Acknowledgements

Our work is partly supported by Ministry of Education and Science of the Russian Federation under contract 14.740.11.0081, by NSh-3349.2012.2, by RFBR grants 10-02-00509 (A.Mir.), 10-02-00499 (A.Mor.), 11-02-01220 (And.Mor.) and by joint grants 11-02-90453-Ukr, 12-02-91000-ANF, 11-01-92612-Royal Society. The research of H. I. and A.Mir. is supported in part by the Grant-in-Aid for Scientific Research (23540316) from the Ministry of Education, Science and Culture, Japan, and that of A.Mor. by JSPS Invitation Fellowship Program for Research in Japan (S-11137). Support from JSPS/RFBR bilateral collaboration "Synthesis of integrabilities arising from gauge-string duality" (FY2010-2011: 12-02-92108-Yaf-a) is gratefully appreciated.

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Appendix. Tables of polynomials

In this Appendix we list the HOMFLY polynomials and related quantities for all 3-strand knots in the fundamental, symmetric [2] and antisymmetric [1, 1] representations. Namely, for each knot, besides HOMFLY, we write down expressions for the Jones ($A = q^2$), Alexander ($A = 1$), "special" ($q \rightarrow 1$) polynomials, the Ooguri-Vafa polynomials $f_R(q, A)$ and their "special" limit $q \rightarrow 1$, and for the numbers $N_{R,n,k}$ in (68). Note that all the expressions are listed with the factors (82) taken into account. We also use the notation $\{x\} \equiv x - x^{-1}$.

Knot 3₁

$$(-1, -1 | -1, -1)$$

We use here the $(-1, -1 | -1, -1)$ representation of the trefoil (in contrast with $(1, 1 | 1, 1)$ used throughout the main body of the paper) in order to match the standard knot tables.

HOMFLY polynomials

$$\frac{H_{[1]}}{*S_{[1]}} = A^4 \left((q^2 + q^{-2})A^{-2} - 1 \right) = \frac{A^4}{*S_{[1]}} \left(*S_{[3]}q^{-4} - *S_{[2,1]} + *S_{[1,1,1]}q^4 \right) \quad (103)$$

$$\begin{aligned} \frac{H_{[1,1]}}{*S_{[1,1]}} &= A^8 q^{-16} \left((q^{20} + q^{14} + q^{12} + q^8)A^{-4} + (-q^{16} - q^{14} - q^{10} - q^8)A^{-2} + q^{10} \right) = \\ &= \frac{A^8 q^{-16}}{*S_{[1,1]}} \left(*S_{[3,3]} - *S_{[3,2,1]}q^4 + *S_{[3,1,1,1]}q^8 + *S_{[2,2,2]}q^8 - *S_{[2,1,1,1,1]}q^{16} + *S_{[1,1,1,1,1,1]}q^{24} \right) \end{aligned} \quad (104)$$

$$\begin{aligned} \frac{H_{[2]}}{*S_{[2]}} &= A^8 q^{16} \left((q^{-8} + q^{-12} + q^{-14} + q^{-20})A^{-4} + (-q^{-8} - q^{-10} - q^{-14} - q^{-16})A^{-2} + q^{-10} \right) = \\ &= \frac{A^8 q^{16}}{*S_{[2]}} \left(*S_{[6]}q^{-24} - *S_{[5,1]}q^{-16} + *S_{[4,1,1]}q^{-8} + *S_{[3,3]}q^{-8} - *S_{[3,2,1]}q^{-4} + *S_{[2,2,2]} \right) \end{aligned} \quad (105)$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = q^2 - 1 + q^{-2} \quad (106)$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = q^4 - 1 + q^{-4} \quad (107)$$

Jones polynomials

$$J_{[1]} = -q^8 + q^6 + q^2 \quad (108)$$

$$J_{[1,1]} = 1 \quad (109)$$

$$J_{[2]} = q^{22} - q^{20} - q^{18} + q^{16} - q^{14} + q^{10} + q^4 \quad (110)$$

Special polynomials

$$\mathfrak{H}_{[1]} = 2A^2 - A^4 \quad (111)$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 4A^4 - 4A^6 + A^8 \quad (112)$$

Ooguri-Vafa polynomials

$$f_{[1,1]} = -\frac{A^6 \{A\}^2 \{A/q\} \{Aq\} (q^2 + q^{-2})}{q^3 \{q\}} \quad (113)$$

$$f_{[2]} = \frac{q^3 A^6 \{A\}^2 \{A/q\} \{Aq\} (q^2 + q^{-2})}{\{q\}} \quad (114)$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = 2A^6(A - A^{-1})^3 \quad (115)$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = (A^2 - A^{-2})(A - A^{-1})^3 \quad (128)$$

Numbers $N_{R,n,k}$

$k \setminus n =$	-3	-1	1	3
$N_{[1]} :$				
0	-1	2	-2	1
1	0	1	-1	0

$k \setminus n =$	-6	-4	-2	0	2	4	6
$N_{[1,1]} :$							
0	1	-3	4	-6	9	-7	2
1	0	-1	2	-5	9	-6	1
2	0	0	0	-1	2	-1	0

$k \setminus n =$	-6	-4	-2	0	2	4	6
$N_{[2]} :$							
0	2	-7	9	-6	4	-3	1
1	1	-6	9	-5	2	-1	0
2	0	-1	2	-1	0	0	0

Knot 5_2

$$(1, -1|1, 3)$$

The choice of representation for this knot is different in [29] and in [9]. We here follow the convention of [9].

HOMFLY polynomials

$$\begin{aligned} \frac{H_{[1]}}{*S_{[1]}} &= A^{-4} \left((q^2 - 1 + q^{-2})A^{-2} + (q^2 - 1 + q^{-2}) - A^2 \right) = \\ &= \frac{A^{-4}}{*S_{[1]}} \left(*S_{[3]}q^4 + *S_{[2,1]}(-q^4 + q^2 - 1 + q^{-2} - q^{-4}) + *S_{[1,1,1]}q^{-4} \right) \end{aligned} \quad (129)$$

$$\begin{aligned} \frac{H_{[1,1]}}{*S_{[1,1]}} &= q^{16}A^{-8} \left((q^{20} - q^{18} - q^{16} + 2q^{14} - q^{10} + q^8)A^{-4} + (q^{20} + q^{18} - 2q^{16} + 3q^{12} - q^{10} - q^8 + q^6)A^{-2} + \right. \\ &\quad \left. + (-2q^{14} + 2q^{10} - q^8 - q^6 + q^4) + (-q^{12} + q^8 - q^6 - q^4)A^2 + q^6A^4 \right) = \\ &= \frac{q^{16}A^{-8}}{*S_{[1,1]}} \left(*S_{[3,3]}1 + *S_{[3,2,1]}(-1 + q^{-2} - q^{-4} + q^{-6} - q^{-8}) + *S_{[3,1,1,1]}q^{-8} + *S_{[2,2,2]}q^{-8} + \right. \\ &\quad \left. + *S_{[2,2,1,1]}(q^2 - 1 - q^{-2} + 2q^{-4} - 2q^{-8} + q^{-10} + q^{-12} - 2q^{-14} + q^{-16} - q^{-20} + q^{-22}) + \right. \\ &\quad \left. + *S_{[2,1,1,1,1]}(-q^{-8} + q^{-12} - q^{-16} + q^{-20} - q^{-24}) + *S_{[1,1,1,1,1,1]}q^{-24} \right) \end{aligned} \quad (130)$$

$$\begin{aligned} \frac{H_{[2]}}{*S_{[2]}} &= q^{-16}A^{-8} \left((q^{-8} - q^{-10} + 2q^{-14} - q^{-16} - q^{-18} + q^{-20})A^{-4} + \right. \\ &\quad \left. + (q^{-6} - q^{-8} - q^{-10} + 3q^{-12} - 2q^{-16} + q^{-18} + q^{-20})A^{-2} + \right. \\ &\quad \left. + (q^{-4} - q^{-6} - q^{-8} + 2q^{-10} - 2q^{-14}) + (-q^{-4} - q^{-6} + q^{-8} - q^{-12})A^2 + q^{-6}A^4 \right) = \\ &= \frac{q^{-16}A^{-8}}{*S_{[2]}} \left(*S_{[6]}q^{24} + *S_{[5,1]}(-q^{24} + q^{20} - q^{16} + q^{12} - q^8) + \right. \\ &\quad \left. + *S_{[4,2]}(q^{22} - q^{20} + q^{16} - 2q^{14} + q^{12} + q^{10} - 2q^8 + 2q^4 - q^2 - 1 + q^{-2}) + \right. \\ &\quad \left. + *S_{[4,1,1]}q^8 + *S_{[3,3]}q^8 + *S_{[3,2,1]}(-q^8 + q^6 - q^4 + q^2 - 1) + *S_{[2,2,2]} \right) \end{aligned} \quad (131)$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = 2q^2 - 3 + 2q^{-2} \quad (132)$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = 2q^4 - 3 + 2q^{-4} \quad (133)$$

Jones polynomials

$$J_{[1]} = q^{-2} - q^{-4} + 2q^{-6} - q^{-8} + q^{-10} - q^{-12} \quad (134)$$

$$J_{[1,1]} = 1 \quad (135)$$

$$J_{[2]} = q^{-4} - q^{-6} + 3q^{-10} - 2q^{-12} - q^{-14} + 4q^{-16} - 3q^{-18} - q^{-20} + 3q^{-22} - 2q^{-24} - q^{-26} + 2q^{-28} - q^{-30} - q^{-32} + q^{-34} \quad (136)$$

Special polynomials

$$\mathfrak{H}_{[1]} = -A^{-6} + A^{-4} + A^{-2} \quad (137)$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = A^{-12} - 2A^{-10} - A^{-8} + 2A^{-6} + A^{-4} \quad (138)$$

Ooguri-Vafa polynomials

$$f_{[1,1]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2}) \left((q^9 + q^7 + q^5)A^{-10} + (q^7 + q^5)A^{-8} + q^5 A^{-6} \right)}{\{q\}} \quad (139)$$

$$f_{[2]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2}) \left((-q^{-1} - q^{-5} - q^{-9})A^{-10} + (-q^{-1} - q^{-7})A^{-8} + (-q^{-1} + q^{-3} - q^{-5})A^{-6} \right)}{\{q\}} \quad (140)$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = -\frac{(A^4 + 2A^2 + 3)(A - A^{-1})^3}{A^{10}} \quad (141)$$

Numbers $N_{R,n,k}$

$N_{[1]} :$	$k \setminus n =$	-7	-5	-3	-1
	0	1	-2	0	1
	1	0	-1	0	1

$N_{[1,1]} :$	$k \setminus n =$	-14	-12	-10	-8	-6	-4	-2
	0	6	-21	25	-10	0	-1	1
	1	11	-46	60	-25	0	-1	1
	2	6	-34	50	-22	0	0	0
	3	1	-10	17	-8	0	0	0
	4	0	-1	2	-1	0	0	0

$N_{[2]} :$	$k \setminus n =$	-14	-12	-10	-8	-6	-4	-2
	0	9	-31	36	-14	0	-3	2
	1	24	-95	120	-49	0	-4	3
	2	22	-106	147	-63	0	-1	1
	3	8	-53	82	-37	0	0	0
	4	1	-12	21	-10	0	0	0
	5	0	-1	2	-1	0	0	0

Knot 6₂

$$(1, -1|1, -3)$$

HOMFLY polynomials

$$\begin{aligned} \frac{H_{[1]}}{*S_{[1]}} &= A^2 \left((q^2 + q^{-2})A^{-2} + (-q^4 + q^2 - 2 + q^{-2} - q^{-4}) + (q^2 - 1 + q^{-2})A^2 \right) = \\ &= \frac{A^2}{*S_{[1]}} \left(*S_{[3]}q^{-2} + *S_{[2,1]}(q^6 - 2q^4 + 2q^2 - 3 + 2q^{-2} - 2q^{-4} + q^{-6}) + *S_{[1,1,1]}q^2 \right) \end{aligned} \quad (142)$$

$$\begin{aligned} \frac{H_{[1,1]}}{*S_{[1,1]}} &= A^4 q^{-8} \left((q^{16} + q^{10} + q^8 + q^4)A^{-4} + (-q^{18} - q^{16} + q^{14} - q^{12} - 3q^{10} - 2q^4 - q^{-2})A^{-2} + \right. \\ &\quad \left. + (q^{18} - q^{16} + 4q^{12} - 3q^8 + 4q^6 + 2q^4 - 2q^2 + 2 + q^{-2} - q^{-4} + q^{-6}) + \right. \\ &\quad \left. + (-q^{14} + 2q^{10} - 2q^8 - 3q^6 + 3q^4 - 3 + q^{-2} - q^{-6})A^2 + (q^8 - q^6 - q^4 + 2q^2 - q^{-2} + q^{-4})A^4 \right) = \\ &= \frac{A^4 q^{-8}}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(q^8 - 2q^6 + 2q^4 - 3q^2 + 2 - 2q^{-2} + q^{-4}) + *S_{[3,1,1,1]}q^4 + *S_{[2,2,2]}q^4 + \right. \end{aligned} \quad (143)$$

$$\begin{aligned} &\quad + *S_{[2,2,1,1]}(q^{24} - 2q^{22} - q^{20} + 5q^{18} - 3q^{16} - 5q^{14} + 8q^{12} - 9q^8 + \\ &\quad + 7q^6 + 3q^4 - 8q^2 + 5 + 2q^{-2} - 5q^{-4} + 3q^{-6} - 2q^{-10} + q^{-12}) + \\ &\quad + *S_{[2,1,1,1,1]}(q^{20} - 2q^{16} + 2q^{12} - 3q^8 + 2q^4 - 2 + q^{-4}) + *S_{[1,1,1,1,1,1]}q^{12} \Big) \\ \frac{H_{[2]}}{*S_{[2]}} &= A^4 q^8 \left((q^{-4} + q^{-8} + q^{-10} + q^{-16})A^{-4} + (-q^2 - 2q^{-4} - 3q^{-10} - q^{-12} + q^{-14} - q^{-16} - q^{-18})A^{-2} + \right. \\ &\quad \left. + (q^6 - q^4 + q^2 + 2 - 2q^{-2} + 2q^{-4} + 4q^{-6} - 3q^{-8} + 4q^{-12} - q^{-16} + q^{-18}) + \right. \\ &\quad \left. + (-q^6 + q^2 - 3 + 3q^{-4} - 3q^{-6} - 2q^{-8} + 2q^{-10} - q^{-14})A^2 + (q^4 - q^2 + 2q^{-2} - q^{-4} - q^{-6} + q^{-8})A^4 \right) = \\ &= \frac{A^4 q^8}{*S_{[2]}} \left(*S_{[6]}q^{-12} + *S_{[5,1]}(q^4 - 2 + 2q^{-4} - 3q^{-8} + 2q^{-12} - 2q^{-16} + q^{-20}) + \right. \\ &\quad + *S_{[4,2]}(q^{12} - 2q^{10} + 3q^6 - 5q^4 + 2q^2 + 5 - 8q^{-2} + 3q^{-4} + \\ &\quad + 7q^{-6} - 9q^{-8} + 8q^{-12} - 5q^{-14} - 3q^{-16} + 5q^{-18} - q^{-20} - 2q^{-22} + q^{-24}) + \\ &\quad \left. + *S_{[4,1,1]}q^{-4} + *S_{[3,3]}q^{-4} + *S_{[3,2,1]}(q^4 - 2q^2 + 2 - 3q^{-2} + 2q^{-4} - 2q^{-6} + q^{-8}) + *S_{[2,2,2]} \right) \end{aligned} \quad (144)$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = -q^4 + 3q^2 - 3 + 3q^{-2} - q^{-4} \quad (145)$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = -q^8 + 3q^4 - 3 + 3q^{-4} - q^{-8} \quad (146)$$

Jones polynomials

$$J_{[1]} = q^{10} - 2q^8 + 2q^6 - 2q^4 + 2q^2 - 1 + q^{-2} \quad (147)$$

$$J_{[1,1]} = 1 \quad (148)$$

$$J_{[2]} = q^{28} - 2q^{26} + 4q^{22} - 5q^{20} + 6q^{16} - 6q^{14} + 6q^{10} - 5q^8 - q^6 + 5q^4 - 3q^2 - 1 + 3q^{-2} - q^{-4} - q^{-6} + q^{-8} \quad (149)$$

Special polynomials

$$\mathfrak{H}_{[1]} = 2 - 2A^2 + A^4 \quad (150)$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 4 - 8A^2 + 8A^4 - 4A^6 + A^8 \quad (151)$$

Ooguri-Vafa polynomials

$$f_{[1,1]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2}) ((q^7 + q^5)A^2 + (q^5 + q^{-3})A^4 + (-q^{-1} - q^{-7})A^6)}{\{q\}} \quad (152)$$

$$f_{[2]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2}) ((-q^{-1} - q^{-3})A^2 + (-q^7 - q^{-1})A^4 + (q^{11} + q^5)A^6)}{\{q\}} \quad (153)$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = 2A^2(A^4 - A^2 - 1)(A - A^{-1})^3 \quad (154)$$

Numbers $N_{R,n,k}$

$N_{[1]} :$

$k \backslash n =$	-1	1	3	5
0	-2	4	-3	1
1	-1	4	-4	1
2	0	1	-1	0

$N_{[1,1]} :$

$k \backslash n =$	-2	0	2	4	6	8	10
0	3	-14	33	-52	53	-30	7
1	4	-25	84	-178	212	-125	28
2	1	-13	82	-246	335	-201	42
3	0	-2	40	-175	267	-159	29
4	0	0	10	-67	113	-65	9
5	0	0	1	-13	24	-13	1
6	0	0	0	-1	2	-1	0

$N_{[2]} :$

$k \backslash n =$	-2	0	2	4	6	8	10
0	5	-20	35	-40	35	-20	5
1	10	-45	85	-110	110	-65	15
2	6	-34	72	-113	132	-79	16
3	1	-10	25	-54	75	-44	7
4	0	-1	3	-12	20	-11	1
5	0	0	0	-1	2	-1	0

Knot 6_3

$(2, -1|1, -2)$

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= (-q^2 + 1 - q^{-2})A^{-2} + (q^4 - q^2 + 3 - q^{-2} + q^{-4}) + (-q^2 + 1 - q^{-2})A^2 = \\
&= \frac{1}{*S_{[1]}} \left(*S_{[3]} + *S_{[2,1]}(-q^6 + 2q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} - q^{-6}) + *S_{[1,1,1]} \right)
\end{aligned} \tag{155}$$

$$\begin{aligned}
\frac{H_{[1,1]}}{*S_{[1,1]}} &= (q^{10} - q^8 - q^6 + 2q^4 - 1 + q^{-2})A^{-4} + (-q^{12} + 2q^8 - 3q^6 - 3q^4 + 4q^2 - 1 - 4q^{-2} + q^{-4} - q^{-8})A^{-2} + \\
&\quad + (q^{12} - q^{10} + q^8 + 4q^6 - 3q^4 - q^2 + 9 - q^{-2} - 3q^{-4} + 4q^{-6} + q^{-8} - q^{-10} + q^{-12}) + \\
&\quad + (-q^8 + q^4 - 4q^2 - 1 + 4q^{-2} - 3q^{-4} - 3q^{-6} + 2q^{-8} - q^{-12})A^2 + (q^2 - 1 + 2q^{-4} - q^{-6} - q^{-8} + q^{-10})A^4 = \\
&= \frac{1}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(-q^6 + 2q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} - q^{-6}) + *S_{[3,1,1,1]} + *S_{[2,2,2]} + \right. \\
&\quad + *S_{[2,2,1,1]}(q^{18} - 2q^{16} + 5q^{12} - 6q^{10} - 2q^8 + 12q^6 - 9q^4 - 7q^2 + \\
&\quad + 16 - 7q^{-2} - 9q^{-4} + 12q^{-6} - 2q^{-8} - 6q^{-10} + 5q^{-12} - 2q^{-16} + q^{-18}) + \\
&\quad \left. + *S_{[2,1,1,1,1]}(-q^{12} + 2q^8 - 3q^4 + 3 - 3q^{-4} + 2q^{-8} - q^{-12}) + *S_{[1,1,1,1,1,1]} \right) \\
\frac{H_{[2]}}{*S_{[2]}} &= q^2 - 1 + 2q^{-4} - q^{-6} - q^{-8} + q^{-10})A^{-4} + (-q^8 + q^4 - 4q^2 - 1 + 4q^{-2} - 3q^{-4} - 3q^{-6} + 2q^{-8} - q^{-12})A^{-2} + \quad (156) \\
&\quad + (q^{12} - q^{10} + q^8 + 4q^6 - 3q^4 - q^2 + 9 - q^{-2} - 3q^{-4} + 4q^{-6} + q^{-8} - q^{-10} + q^{-12}) + \\
&\quad + (-q^{12} + 2q^8 - 3q^6 - 3q^4 + 4q^2 - 1 - 4q^{-2} + q^{-4} - q^{-8})A^2 + (q^{10} - q^8 - q^6 + 2q^4 - 1 + q^{-2})A^4 = \\
&= \frac{1}{*S_{[2]}} \left(*S_{[6]} + *S_{[5,1]}(-q^{12} + 2q^8 - 3q^4 + 3 - 3q^{-4} + 2q^{-8} - q^{-12}) + \right. \\
&\quad + *S_{[4,2]}(q^{18} - 2q^{16} + 5q^{12} - 6q^{10} - 2q^8 + 12q^6 - 9q^4 - 7q^2 + \\
&\quad + 16 - 7q^{-2} - 9q^{-4} + 12q^{-6} - 2q^{-8} - 6q^{-10} + 5q^{-12} - 2q^{-16} + q^{-18}) + \\
&\quad + *S_{[4,1,1]} + *S_{[3,3]} + \\
&\quad \left. + *S_{[3,2,1]}(-q^6 + 2q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} - q^{-6}) + *S_{[2,2,2]} \right)
\end{aligned}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = q^4 - 3q^2 + 5 - 3q^{-2} + q^{-4} \quad (157)$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = q^8 - 3q^4 + 5 - 3q^{-4} + q^{-8} \quad (158)$$

Jones polynomials

$$J_{[1]} = -q^6 + 2q^4 - 2q^2 + 3 - 2q^{-2} + 2q^{-4} - q^{-6} \quad (159)$$

$$J_{[1,1]} = 1 \quad (160)$$

$$\begin{aligned}
J_{[2]} &= q^{18} - 2q^{16} - q^{14} + 5q^{12} - 4q^{10} - 3q^8 + 9q^6 - 5q^4 - 5q^2 + \\
&\quad + 11 - 5q^{-2} - 5q^{-4} + 9q^{-6} - 3q^{-8} - 4q^{-10} + 5q^{-12} - q^{-14} - 2q^{-16} + q^{-18}
\end{aligned} \quad (161)$$

Special polynomials

$$\mathfrak{H}_{[1]} = -A^{-2} + 3 - A^2 \quad (162)$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = A^{-4} - 6A^{-2} + 11 - 6A^2 + A^4 \quad (163)$$

Ooguri-Vafa polynomials

$$f_{[1,1]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2}) \left((-q^7 + q^3 - q^{-1})A^{-2} + (q^3 + 2q^1 - 2q^{-1} - q^{-3}) + (q^1 - q^{-3} + q^{-7})A^2 \right)}{\{q\}} \quad (164)$$

$$f_{[2]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2}) \left((-q^1 + q^{-3} - q^{-7})A^{-2} + (-q^3 - 2q^1 + 2q^{-1} + q^{-3}) + (q^7 - q^3 + q^{-1})A^2 \right)}{\{q\}} \quad (165)$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = (A^2 - A^{-2})(A - A^{-1})^3 \quad (166)$$

Numbers $N_{R,n,k}$

$k \setminus n =$	-3	-1	1	3
$N_{[1]} :$				
0	1	-4	4	-1
1	1	-4	4	-1
2	0	-1	1	0

$k \setminus n =$	-6	-4	-2	0	2	4	6
$N_{[1,1]} :$							
0	1	-9	28	-42	33	-13	2
1	5	-27	72	-113	104	-52	11
2	5	-26	65	-114	127	-72	15
3	1	-9	24	-54	74	-43	7
4	0	-1	3	-12	20	-11	1
5	0	0	0	-1	2	-1	0

$k \setminus n =$	-6	-4	-2	0	2	4	6
$N_{[2]} :$							
0	2	-13	33	-42	28	-9	1
1	11	-52	104	-113	72	-27	5
2	15	-72	127	-114	65	-26	5
3	7	-43	74	-54	24	-9	1
4	1	-11	20	-12	3	-1	0
5	0	-1	2	-1	0	0	0

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$(1, -1|1, 5)$

HOMFLY polynomials

$$\begin{aligned} \frac{H_{[1]}}{*S_{[1]}} &= A^{-6} \left((-q^2 - q^{-2})A^{-2} + (q^4 - q^2 + 2 - q^{-2} + q^{-4}) + (q^4 - q^2 + 1 - q^{-2} + q^{-4})A^2 \right) = \\ &= \frac{A^{-6}}{*S_{[1]}} \left(*S_{[3]}q^6 + *S_{[2,1]}(-q^6 + q^4 - 2q^2 + 3 - 2q^{-2} + q^{-4} - q^{-6}) + *S_{[1,1,1]}q^{-6} \right) \end{aligned} \quad (167)$$

$$\begin{aligned} \frac{H_{[1,1]}}{*S_{[1,1]}} &= A^{-12}q^{24} \left((q^{-6} + q^{-10} + q^{-12} + q^{-18})A^{-4} + \right. \\ &\quad + (-q^{-4} - q^{-6} - 2q^{-10} - q^{-12} + q^{-14} - 2q^{-16} - 2q^{-18} + q^{-20} - q^{-24})A^{-2} + \\ &\quad + (q^{-4} - q^{-6} + 2q^{-10} - 2q^{-12} + 4q^{-16} - 3q^{-18} - 2q^{-20} + 3q^{-22} - 2q^{-26}) + \\ &\quad + (q^{-6} - q^{-8} + 2q^{-12} - 2q^{-14} + q^{-16} + 3q^{-18} - 3q^{-20} - q^{-22} + 4q^{-24} - 2q^{-28} + q^{-30} + q^{-32})A^2 + \\ &\quad \left. + (q^{-8} - q^{-10} + q^{-14} - q^{-16} + q^{-18} + q^{-20} - 2q^{-22} + 2q^{-26} - q^{-28} - q^{-30} + q^{-32})A^4 \right) = \\ &= \frac{A^{-12}q^{24}}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(-1 + q^{-2} - 2q^{-4} + 3q^{-6} - 2q^{-8} + q^{-10} - q^{-12}) + *S_{[3,1,1,1]}q^{-12} + *S_{[2,2,2]}q^{-12} + \right. \\ &\quad + *S_{[2,2,1,1]}(q^2 - 1 + 2q^{-4} - 3q^{-6} - q^{-8} + 6q^{-10} - 5q^{-12} - 4q^{-14} + \\ &\quad + 10q^{-16} - 4q^{-18} - 5q^{-20} + 6q^{-22} - q^{-24} - 3q^{-26} + 2q^{-28} - q^{-32} + q^{-34}) + \\ &\quad \left. + *S_{[2,1,1,1,1]}(-q^{-12} + q^{-16} - 2q^{-20} + 3q^{-24} - 2q^{-28} + q^{-32} - q^{-36}) + *S_{[1,1,1,1,1,1]}q^{-36} \right) \end{aligned} \quad (168)$$

$$\begin{aligned}
\frac{H_{[2]}}{*S_{[2]}} &= A^{-12}q^{-24} \left((q^{18} + q^{12} + q^{10} + q^6)A^{-4} + \right. \\
&\quad + (-q^{24} + q^{20} - 2q^{18} - 2q^{16} + q^{14} - q^{12} - 2q^{10} - q^6 - q^4)A^{-2} + \\
&\quad + (-2q^{26} + 3q^{22} - 2q^{20} - 3q^{18} + 4q^{16} - 2q^{12} + 2q^{10} - q^6 + q^4) + \\
&\quad + (q^{32} + q^{30} - 2q^{28} + 4q^{24} - q^{22} - 3q^{20} + 3q^{18} + q^{16} - 2q^{14} + 2q^{12} - q^8 + q^6)A^2 + \\
&\quad \left. + (q^{32} - q^{30} - q^{28} + 2q^{26} - 2q^{22} + q^{20} + q^{18} - q^{16} + q^{14} - q^{10} + q^8)A^4 \right) = \\
&= \frac{A^{-12}q^{-24}}{*S_{[2]}} \left(*S_{[6]}q^{36} + *S_{[5,1]}(-q^{36} + q^{32} - 2q^{28} + 3q^{24} - 2q^{20} + q^{16} - q^{12}) + \right. \\
&\quad + *S_{[4,2]}(q^{34} - q^{32} + 2q^{28} - 3q^{26} - q^{24} + 6q^{22} - 5q^{20} - \\
&\quad - 4q^{18} + 10q^{16} - 4q^{14} - 5q^{12} + 6q^{10} - q^8 - 3q^6 + 2q^4 - 1 + q^{-2}) + \\
&\quad \left. + *S_{[4,1,1]}q^{12} + *S_{[3,3]}q^{12} + *S_{[3,2,1]}(-q^{12} + q^{10} - 2q^8 + 3q^6 - 2q^4 + q^2 - 1) + *S_{[2,2,2]} \right)
\end{aligned} \tag{169}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = 2q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} \tag{170}$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = 2q^8 - 3q^4 + 3 - 3q^{-4} + 2q^{-8} \tag{171}$$

Jones polynomials

$$J_{[1]} = q^{-4} - q^{-6} + 2q^{-8} - 2q^{-10} + 3q^{-12} - 2q^{-14} + q^{-16} - q^{-18} \tag{172}$$

$$J_{[1,1]} = 1 \tag{173}$$

$$\begin{aligned}
J_{[2]} &= q^{-8} - q^{-10} + 3q^{-14} - 2q^{-16} - 2q^{-18} + 5q^{-20} - 2q^{-22} - 4q^{-24} + 7q^{-26} - 2q^{-28} - 6q^{-30} + \\
&\quad + 8q^{-32} - 2q^{-34} - 5q^{-36} + 5q^{-38} - q^{-40} - 3q^{-42} + 2q^{-44} - q^{-48} + q^{-50}
\end{aligned} \tag{174}$$

Special polynomials

$$\mathfrak{H}_{[1]} = -2A^{-8} + 2A^{-6} + A^{-4} \tag{175}$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 4A^{-16} - 8A^{-14} + 4A^{-10} + A^{-8} \tag{176}$$

Ooguri-Vafa polynomials

$$\begin{aligned}
f_{[1,1]} &= \frac{\{A\}^2\{A/q\}\{Aq\}}{\{q\}} \left((q^{17} + 2q^{13} + q^{11} + 2q^9 + q^7 + 3q^5 + q^3 + 2q^1 + q^{-3})A^{-14} + \right. \\
&\quad \left. + (q^{15} + q^{11} + q^9 + q^5 + q^3 + q^1 - q^{-1} + q^{-3})A^{-12} + (q^{13} - q^{11} + q^9 + 2q^1 - 2q^{-1} + q^{-3})A^{-10} \right)
\end{aligned} \tag{177}$$

$$\begin{aligned}
f_{[2]} &= \frac{\{A\}^2\{A/q\}\{Aq\}}{\{q\}} \left((-q^3 - 2q^{-1} - q^{-3} - 3q^{-5} - q^{-7} - 2q^{-9} - q^{-11} - 2q^{-13} - q^{-17})A^{-14} + \right. \\
&\quad \left. + (-q^3 + q^1 - q^{-1} - q^{-3} - q^{-5} - q^{-9} - q^{-11} - q^{-15})A^{-12} + (-q^3 + 2q^1 - 2q^{-1} - q^{-9} + q^{-11} - q^{-13})A^{-10} \right)
\end{aligned} \tag{178}$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = -\frac{2(A^4 + 3A^2 + 7)(A - A^{-1})^3}{A^{14}} \tag{179}$$

Numbers $N_{R,n,k}$

$k \setminus n =$	-9	-7	-5	-3
0	2	-4	1	1
1	1	-4	0	3
2	0	-1	0	1

$N_{[1]} :$

$N_{[1,1]} :$

$k \setminus n =$	-18	-16	-14	-12	-10	-8	-6
0	41	-146	179	-76	-1	-2	5
1	200	-755	944	-390	-8	-15	24
2	398	-1639	2131	-890	-6	-35	41
3	412	-1917	2628	-1123	-1	-28	29
4	241	-1320	1926	-847	0	-9	9
5	80	-549	859	-390	0	-1	1
6	14	-135	228	-107	0	0	0
7	1	-18	33	-16	0	0	0
8	0	-1	2	-1	0	0	0

$k \setminus n =$	-18	-16	-14	-12	-10	-8	-6
0	55	-196	241	-104	1	-4	7
1	330	-1231	1531	-634	-10	-27	41
2	821	-3284	4213	-1751	-15	-75	91
3	1085	-4787	6418	-2716	-7	-85	92
4	837	-4202	5940	-2575	-1	-45	46
5	389	-2314	3472	-1547	0	-11	11
6	107	-803	1286	-590	0	-1	1
7	16	-170	292	-138	0	0	0
8	1	-20	37	-18	0	0	0
9	0	-1	2	-1	0	0	0

$N_{[2]} :$

Knot 7₅

$(-2, 1 | -1, -4)$

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= A^6 \left((q^4 - q^2 + 2 - q^{-2} + q^{-4})A^{-2} + (q^4 - 2q^2 + 2 - 2q^{-2} + q^{-4}) + (-q^2 + 1 - q^{-2})A^2 \right) = \\
&= \frac{A^6}{*S_{[1]}} \left(*S_{[3]}q^{-6} + *S_{[2,1]}(-q^6 + 2q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} - q^{-6}) + *S_{[1,1,1]}q^6 \right)
\end{aligned}$$

(180)

$$\begin{aligned}
\frac{H_{[1,1]}}{*S_{[1,1]}} &= A^{12}q^{-24} \left((q^{32} - q^{30} - q^{28} + 3q^{26} - 3q^{22} + 3q^{20} + q^{18} - 2q^{16} + 2q^{14} + q^{12} - q^{10} + q^8)A^{-4} + \right. \\
&\quad + (q^{32} + q^{30} - 3q^{28} + 6q^{24} - 3q^{22} - 5q^{20} + 6q^{18} - 5q^{14} + 3q^{12} - 2q^8 + q^6)A^{-2} + \\
&\quad + (-3q^{26} + 5q^{22} - 5q^{20} - 5q^{18} + 7q^{16} - q^{14} - 5q^{12} + 4q^{10} - 2q^6 + q^4) + \\
&\quad + (-q^{24} + q^{22} + 3q^{20} - 3q^{18} - 2q^{16} + 5q^{14} - q^{12} - 3q^{10} + 2q^8 - q^4)A^2 + \\
&\quad \left. + (q^{18} - q^{16} - q^{14} + 2q^{12} - q^8 + q^6)A^4 \right) = \\
&= \frac{A^{12}q^{-24}}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(-q^{12} + 2q^{10} - 3q^8 + 3q^6 - 3q^4 + 2q^2 - 1) + *S_{[3,1,1,1]}q^{12} + *S_{[2,2,2]}q^{12} + \right. \\
&\quad + *S_{[2,2,1,1]}(q^{34} - 2q^{32} + 5q^{28} - 6q^{26} - 2q^{24} + 12q^{22} - 9q^{20} - \\
&\quad - 7q^{18} + 16q^{16} - 7q^{14} - 9q^{12} + 12q^{10} - 2q^8 - 6q^6 + 5q^4 - 2 + q^{-2}) + \\
&\quad \left. + *S_{[2,1,1,1,1]}(-q^{36} + 2q^{32} - 3q^{28} + 3q^{24} - 3q^{20} + 2q^{16} - q^{12}) + *S_{[1,1,1,1,1,1]}q^{36} \right) \\
\frac{H_{[2]}}{*S_{[2]}} &= A^{12}q^{24} \left((q^{-8} - q^{-10} + q^{-12} + 2q^{-14} - 2q^{-16} + q^{-18} + 3q^{-20} - 3q^{-22} + 3q^{-26} - q^{-28} - q^{-30} + q^{-32})A^{-4} + \right. \\
&\quad + (q^{-6} - 2q^{-8} + 3q^{-12} - 5q^{-14} + 6q^{-18} - 5q^{-20} - 3q^{-22} + 6q^{-24} - 3q^{-28} + q^{-30} + q^{-32})A^{-2} + \\
&\quad + (q^{-4} - 2q^{-6} + 4q^{-10} - 5q^{-12} - q^{-14} + 7q^{-16} - 5q^{-18} - 5q^{-20} + 5q^{-22} - 3q^{-26}) + \\
&\quad + (-q^{-4} + 2q^{-8} - 3q^{-10} - q^{-12} + 5q^{-14} - 2q^{-16} - 3q^{-18} + 3q^{-20} + q^{-22} - q^{-24})A^2 + \\
&\quad + (q^{-6} - q^{-8} + 2q^{-12} - q^{-14} - q^{-16} + q^{-18})A^4 \Big) = \\
&= \frac{A^{12}q^{24}}{*S_{[2]}} \left(*S_{[6]}q^{-36} + *S_{[5,1]}(-q^{-12} + 2q^{-16} - 3q^{-20} + 3q^{-24} - 3q^{-28} + 2q^{-32} - q^{-36}) + \right. \\
&\quad + *S_{[4,2]}(q^2 - 2 + 5q^{-4} - 6q^{-6} - 2q^{-8} + 12q^{-10} - 9q^{-12} - 7q^{-14} + \\
&\quad + 16q^{-16} - 7q^{-18} - 9q^{-20} + 12q^{-22} - 2q^{-24} - 6q^{-26} + 5q^{-28} - 2q^{-32} + q^{-34}) + \\
&\quad \left. + *S_{[4,1,1]}q^{-12} + *S_{[3,3]}q^{-12} + *S_{[3,2,1]}(-1 + 2q^{-2} - 3q^{-4} + 3q^{-6} - 3q^{-8} + 2q^{-10} - q^{-12}) + *S_{[2,2,2]} \right)
\end{aligned} \tag{181}$$

$$\begin{aligned}
&= \frac{A^{12}q^{24}}{*S_{[2]}} \left(*S_{[6]}q^{-36} + *S_{[5,1]}(-q^{-12} + 2q^{-16} - 3q^{-20} + 3q^{-24} - 3q^{-28} + 2q^{-32} - q^{-36}) + \right. \\
&\quad + *S_{[4,2]}(q^2 - 2 + 5q^{-4} - 6q^{-6} - 2q^{-8} + 12q^{-10} - 9q^{-12} - 7q^{-14} + \\
&\quad + 16q^{-16} - 7q^{-18} - 9q^{-20} + 12q^{-22} - 2q^{-24} - 6q^{-26} + 5q^{-28} - 2q^{-32} + q^{-34}) + \\
&\quad \left. + *S_{[4,1,1]}q^{-12} + *S_{[3,3]}q^{-12} + *S_{[3,2,1]}(-1 + 2q^{-2} - 3q^{-4} + 3q^{-6} - 3q^{-8} + 2q^{-10} - q^{-12}) + *S_{[2,2,2]} \right)
\end{aligned} \tag{182}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = 2q^4 - 4q^2 + 5 - 4q^{-2} + 2q^{-4} \tag{183}$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = 2q^8 - 4q^4 + 5 - 4q^{-4} + 2q^{-8} \tag{184}$$

Jones polynomials

$$J_{[1]} = -q^{18} + 2q^{16} - 3q^{14} + 3q^{12} - 3q^{10} + 3q^8 - q^6 + q^4 \tag{185}$$

$$J_{[1,1]} = 1 \tag{186}$$

$$\begin{aligned}
J_{[2]} &= q^{50} - 2q^{48} + 5q^{44} - 6q^{42} - 2q^{40} + 11q^{38} - 9q^{36} - 4q^{34} + 14q^{32} - \\
&\quad - 10q^{30} - 5q^{28} + 13q^{26} - 7q^{24} - 5q^{22} + 9q^{20} - 3q^{18} - 3q^{16} + 4q^{14} - q^{10} + q^8
\end{aligned} \tag{187}$$

Special polynomials

$$\mathfrak{H}_{[1]} = 2A^4 - A^8 \tag{188}$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 4A^8 - 4A^{12} + A^{16} \tag{189}$$

Ooguri-Vafa polynomials

$$\begin{aligned}
f_{[1,1]} &= \frac{\{A\}^2\{A/q\}\{Aq\}(q^2 - 1 + q^{-2})}{\{q\}} \left((-2q^3 + q^1 - q^{-1} - 2q^{-3} - q^{-5} - q^{-9})A^{10} + \right. \\
&\quad \left. + (-2q^3 - q^{-3} - 2q^{-5} - q^{-11})A^{12} + (-q^3 - q^{-5} - q^{-7} - q^{-13})A^{14} \right)
\end{aligned} \tag{190}$$

$$\begin{aligned}
f_{[2]} &= \frac{\{A\}^2\{A/q\}\{Aq\}(q^2 - 1 + q^{-2})}{\{q\}} \left((q^{13} + q^9 + 2q^7 + q^5 - q^3 + 2q^1)A^{10} + \right. \\
&\quad \left. + (q^{15} + 2q^9 + q^7 + 2q^1)A^{12} + (q^{17} + q^{11} + q^9 + q^1)A^{14} \right)
\end{aligned} \tag{191}$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = 2A^{10}(2A^4 + 3A^2 + 3)(A - A^{-1})^3 \tag{192}$$

Numbers $N_{R,n,k}$

$k \setminus n =$	3	5	7	9
0	-2	2	1	-1
1	-3	1	3	-1
2	-1	0	1	0

$N_{[1]} :$

$N_{[1,1]} :$

$k \setminus n =$	6	8	10	12	14	16	18
0	17	-50	47	-28	47	-50	17
1	71	-203	181	-218	481	-443	131
2	118	-324	269	-736	1764	-1484	393
3	101	-261	197	-1383	3332	-2604	618
4	47	-112	75	-1563	3674	-2681	560
5	11	-24	14	-1091	2480	-1689	299
6	1	-2	1	-470	1032	-654	92
7	0	0	0	-121	257	-151	15
8	0	0	0	-17	35	-19	1
9	0	0	0	-1	2	-1	0

$N_{[2]} :$

$k \setminus n =$	6	8	10	12	14	16	18
0	11	-32	31	-24	41	-40	13
1	37	-103	91	-144	325	-289	83
2	48	-127	101	-400	968	-793	203
3	30	-74	52	-619	1482	-1125	254
4	9	-20	12	-560	1295	-912	176
5	1	-2	1	-299	665	-433	67
6	0	0	0	-92	197	-118	13
7	0	0	0	-15	31	-17	1
8	0	0	0	-1	2	-1	0

Knot 8_2

$(1, -1|1, -5)$

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= A^4 \left((q^4 + 1 + q^{-4})A^{-2} + (-q^6 + q^4 - 2q^2 + 1 - 2q^{-2} + q^{-4} - q^{-6}) + (q^4 - q^2 + 1 - q^{-2} + q^{-4})A^2 \right) = \\
&= \frac{A^4}{*S_{[1]}} \left(*S_{[3]}q^{-4} + *S_{[2,1]}(q^8 - 2q^6 + 2q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} - 2q^{-6} + q^{-8}) + *S_{[1,1,1]}q^4 \right)
\end{aligned} \tag{193}$$

$$\begin{aligned}
\frac{H_{[1,1]}}{*S_{[1,1]}} &= A^8 q^{-16} \left((q^{28} + q^{22} + q^{20} + q^{16} + q^{14} + q^{12} + q^{10} + q^8 + q^4) A^{-4} + \right. \\
&\quad + (-q^{30} - q^{28} + q^{26} - q^{24} - 3q^{22} - 3q^{16} - 2q^{14} - q^{12} - 2q^{10} - q^8 - q^6 - 2q^4 - q^{-2}) A^{-2} + \\
&\quad + (q^{30} - q^{28} + 4q^{24} - 3q^{20} + 4q^{18} + 3q^{16} - 2q^{14} + q^{12} + 3q^{10} + 2q^6 + q^4 - q^2 + 2 + q^{-2} - q^{-4} + q^{-6}) + \\
&\quad + (-q^{26} + 2q^{22} - 2q^{20} - 3q^{18} + 3q^{16} + q^{14} - 4q^{12} + 2q^8 - 2q^6 - 2 + q^{-2} - q^{-6}) A^2 + \\
&\quad \left. + (q^{20} - q^{18} - q^{16} + 2q^{14} - 2q^{10} + q^8 + q^6 - q^4 + q^2 - q^{-2} + q^{-4}) A^4 \right) = \tag{194} \\
&= \frac{A^8 q^{-16}}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]} (q^{12} - 2q^{10} + 2q^8 - 3q^6 + 3q^4 - 3q^2 + 2 - 2q^{-2} + q^{-4}) + *S_{[3,1,1,1]} q^8 + *S_{[2,2,2]} q^8 + \right. \\
&\quad + *S_{[2,2,1,1]} (q^{36} - 2q^{34} - q^{32} + 5q^{30} - 3q^{28} - 5q^{26} + 8q^{24} + q^{22} - 9q^{20} + 5q^{18} + 4q^{16} - 7q^{14} + \\
&\quad + 2q^{12} + 3q^{10} - 4q^8 + 2q^6 + 2q^4 - 4q^2 + 3 + q^{-2} - 4q^{-4} + 3q^{-6} - 2q^{-10} + q^{-12}) + \\
&\quad + *S_{[2,1,1,1,1]} (q^{32} - 2q^{28} + 2q^{24} - 3q^{20} + 3q^{16} - 3q^{12} + 2q^8 - 2q^4 + 1) + *S_{[1,1,1,1,1,1]} q^{24} \Big) \\
\frac{H_{[2]}}{*S_{[2]}} &= A^8 q^{16} \left((q^{-4} + q^{-8} + q^{-10} + q^{-12} + q^{-14} + q^{-16} + q^{-20} + q^{-22} + q^{-28}) A^{-4} + \right. \\
&\quad + (-q^2 - 2q^{-4} - q^{-6} - q^{-8} - 2q^{-10} - q^{-12} - 2q^{-14} - 3q^{-16} - 3q^{-22} - q^{-24} + q^{-26} - q^{-28} - q^{-30}) A^{-2} + \\
&\quad + (q^6 - q^4 + q^2 + 2 - q^{-2} + q^{-4} + 2q^{-6} + 3q^{-10} + q^{-12} - 2q^{-14} + 3q^{-16} + 4q^{-18} - 3q^{-20} + 4q^{-24} - q^{-28} + q^{-30}) + \\
&\quad + (-q^6 + q^2 - 2 - 2q^{-6} + 2q^{-8} - 4q^{-12} + q^{-14} + 3q^{-16} - 3q^{-18} - 2q^{-20} + 2q^{-22} - q^{-26}) A^2 + \\
&\quad \left. + (q^4 - q^2 + q^{-2} - q^{-4} + q^{-6} + q^{-8} - 2q^{-10} + 2q^{-14} - q^{-16} - q^{-18} + q^{-20}) A^4 \right) = \tag{195} \\
&= \frac{A^8 q^{16}}{*S_{[2]}} \left(*S_{[6]} q^{-24} + *S_{[5,1]} (1 - 2q^{-4} + 2q^{-8} - 3q^{-12} + 3q^{-16} - 3q^{-20} + 2q^{-24} - 2q^{-28} + q^{-32}) + \right. \\
&\quad + *S_{[4,2]} (q^{12} - 2q^{10} + 3q^6 - 4q^4 + q^2 + 3 - 4q^{-2} + 2q^{-4} + 2q^{-6} - 4q^{-8} + 3q^{-10} + \\
&\quad + 2q^{-12} - 7q^{-14} + 4q^{-16} + 5q^{-18} - 9q^{-20} + q^{-22} + 8q^{-24} - 5q^{-26} - 3q^{-28} + 5q^{-30} - q^{-32} - 2q^{-34} + q^{-36}) + \\
&\quad \left. + *S_{[4,1,1]} q^{-8} + *S_{[3,3]} q^{-8} + *S_{[3,2,1]} (q^4 - 2q^2 + 2 - 3q^{-2} + 3q^{-4} - 3q^{-6} + 2q^{-8} - 2q^{-10} + q^{-12}) + *S_{[2,2,2]} \right)
\end{aligned}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = -q^6 + 3q^4 - 3q^2 + 3 - 3q^{-2} + 3q^{-4} - q^{-6} \tag{196}$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = -q^{12} + 3q^8 - 3q^4 + 3 - 3q^{-4} + 3q^{-8} - q^{-12} \tag{197}$$

Jones polynomials

$$J_{[1]} = q^{16} - 2q^{14} + 2q^{12} - 3q^{10} + 3q^8 - 2q^6 + 2q^4 - q^2 + 1 \tag{198}$$

$$J_{[1,1]} = 1 \tag{199}$$

$$\begin{aligned}
J_{[2]} &= q^{44} - 2q^{42} + 3q^{38} - 4q^{36} + 2q^{34} + 3q^{32} - 6q^{30} + 3q^{28} + 4q^{26} - 7q^{24} + 2q^{22} + \\
&\quad + 5q^{20} - 7q^{18} + q^{16} + 5q^{14} - 5q^{12} + 5q^8 - 3q^6 - q^4 + 3q^2 - 1 - q^{-2} + q^{-4}
\end{aligned} \tag{200}$$

Special polynomials

$$\mathfrak{H}_{[1]} = 3A^2 - 3A^4 + A^6 \tag{201}$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 9A^4 - 18A^6 + 15 - 6A^{10} + A^{12} \tag{202}$$

Ooguri-Vafa polynomials

$$\begin{aligned}
f_{[1,1]} &= \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2})}{\{q\}} \left((q^{11} + q^9 - q^{-1} - q^{-3} - q^{-5}) A^6 + \right. \\
&\quad \left. + (q^9 + 2q^1 + q^{-7} + q^{-9} + q^{-11}) A^8 + (-q^3 + q^1 - q^{-3} - q^{-5} - q^{-15}) A^{10} \right)
\end{aligned} \tag{203}$$

$$f_{[2]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2}) ((q^9 + q^7 + q^5 - q^{-5} - q^{-7}) A^6 + (-q^{15} - q^{13} - q^{11} - 2q^3 - q^{-5}) A^8 + (q^{19} + q^9 + q^7 - q^3 + q^1) A^{10})}{\{q\}} \tag{204}$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = A^6 (3A^4 - 6A^2 + 1) (A - A^{-1})^3 \tag{205}$$

HOMFLY polynomials

$$\begin{aligned} \frac{H_{[1]}}{*S_{[1]}} &= A^{-4} \left((q^4 - q^2 + 2 - q^{-2} + q^{-4})A^{-2} + (-q^6 + q^4 - 3q^2 + 1 - 3q^{-2} + q^{-4} - q^{-6}) + (q^4 + 2 + q^{-4})A^2 \right) = \\ &= \frac{A^{-4}}{*S_{[1]}} \left(*S_{[3]}q^4 + *S_{[2,1]}(q^8 - 2q^6 + 3q^4 - 4q^2 + 3 - 4q^{-2} + 3q^{-4} - 2q^{-6} + q^{-8}) + *S_{[1,1,1]}q^{-4} \right) \end{aligned} \quad (206)$$

$$\begin{aligned} \frac{H_{[1,1]}}{*S_{[1,1]}} &= A^{-8}q^{16} \left((q^4 - q^2 + 1 + q^{-2} - 2q^{-4} + 2q^{-6} + 2q^{-8} - 3q^{-10} + q^{-12} + 3q^{-14} - q^{-16} - q^{-18} + q^{-20})A^{-4} + \right. \\ &\quad + (-q^6 - 3 + q^{-2} - q^{-4} - 5q^{-6} + 2q^{-8} - q^{-10} - 8q^{-12} + 3q^{-16} - 5q^{-18} - 3q^{-20} + 2q^{-22} - q^{-26})A^{-2} + \\ &\quad + (q^6 - q^4 + 2q^2 + 2 - q^{-2} + 4q^{-4} + 4q^{-6} + 8q^{-10} + 5q^{-12} - \\ &\quad - q^{-14} + 7q^{-16} + 8q^{-18} - 3q^{-20} + q^{-22} + 5q^{-24} - q^{-28} + q^{-30}) + \\ &\quad + (-q^2 - q^{-2} - 3q^{-4} - q^{-6} - 3q^{-8} - 6q^{-10} - 3q^{-12} - 4q^{-14} - \\ &\quad - 7q^{-16} - 2q^{-18} - q^{-20} - 5q^{-22} - 2q^{-24} + q^{-26} - q^{-28} - q^{-30})A^2 + \\ &\quad \left. + (q^{-4} + 2q^{-8} + 2q^{-10} + q^{-12} + 2q^{-14} + 3q^{-16} + 2q^{-20} + 2q^{-22} + q^{-28})A^4 \right) = \end{aligned} \quad (207)$$

$$\begin{aligned} &= \frac{A^{-8}q^{16}}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(q^4 - 2q^2 + 3 - 4q^{-2} + 3q^{-4} - 4q^{-6} + 3q^{-8} - 2q^{-10} + q^{-12}) + *S_{[3,1,1,1]}q^{-8} + \right. \\ &\quad + *S_{[2,2,2]}q^{-8} + *S_{[2,2,1,1]}(q^{12} - 2q^{10} + q^8 + 2q^6 - 6q^4 + 4q^2 + 3 - 8q^{-2} + 6q^{-4} + 2q^{-6} - 9q^{-8} + 8q^{-10} + \\ &\quad + 3q^{-12} - 12q^{-14} + 8q^{-16} + 8q^{-18} - 14q^{-20} + 2q^{-22} + 10q^{-24} - 8q^{-26} - 3q^{-28} + 6q^{-30} - q^{-32} - 2q^{-34} + q^{-36}) + \\ &\quad \left. + *S_{[2,1,1,1,1]}(1 - 2q^{-4} + 3q^{-8} - 4q^{-12} + 3q^{-16} - 4q^{-20} + 3q^{-24} - 2q^{-28} + q^{-32}) + *S_{[1,1,1,1,1,1]}q^{-24} \right) \\ \frac{H_{[2]}}{*S_{[2]}} &= A^{-8}q^{-16} \left((q^{20} - q^{18} - q^{16} + 3q^{14} + q^{12} - 3q^{10} + 2q^8 + 2q^6 - 2q^4 + q^2 + 1 - q^{-2} + q^{-4})A^{-4} + \right. \\ &\quad + (-q^{26} + 2q^{22} - 3q^{20} - 5q^{18} + 3q^{16} - 8q^{12} - q^{10} + 2q^8 - 5q^6 - q^4 + q^2 - 3 - q^{-6})A^{-2} + \\ &\quad + (q^{30} - q^{28} + 5q^{24} + q^{22} - 3q^{20} + 8q^{18} + 7q^{16} - q^{14} + 5q^{12} + 8q^{10} + 4q^6 + 4q^4 - q^2 + 2 + 2q^{-2} - q^{-4} + q^{-6}) + \\ &\quad + (-q^{30} - q^{28} + q^{26} - 2q^{24} - 5q^{22} - q^{20} - 2q^{18} - 7q^{16} - 4q^{14} - 3q^{12} - 6q^{10} - 3q^8 - q^6 - 3q^4 - q^2 - q^{-2})A^2 + \\ &\quad \left. + (q^{28} + 2q^{22} + 2q^{20} + 3q^{16} + 2q^{14} + q^{12} + 2q^{10} + 2q^8 + q^4)A^4 \right) = \end{aligned} \quad (208)$$

$$\begin{aligned} &= \frac{A^{-8}q^{-16}}{*S_{[2]}} \left(*S_{[6]}q^{24} + *S_{[5,1]}(q^{32} - 2q^{28} + 3q^{24} - 4q^{20} + 3q^{16} - 4q^{12} + 3q^8 - 2q^4 + 1) + \right. \\ &\quad + *S_{[4,2]}(q^{36} - 2q^{34} - q^{32} + 6q^{30} - 3q^{28} - 8q^{26} + 10q^{24} + 2q^{22} - 14q^{20} + 8q^{18} + 8q^{16} - 12q^{14} + \\ &\quad + 3q^{12} + 8q^{10} - 9q^8 + 2q^6 + 6q^4 - 8q^2 + 3 + 4q^{-2} - 6q^{-4} + 2q^{-6} + q^{-8} - 2q^{-10} + q^{-12}) + \\ &\quad \left. + *S_{[4,1,1]}q^8 + *S_{[3,3]}q^8 + *S_{[3,2,1]}(q^{12} - 2q^{10} + 3q^8 - 4q^6 + 3q^4 - 4q^2 + 3 - 2q^{-2} + q^{-4}) + *S_{[2,2,2]} \right) \end{aligned}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = -q^6 + 3q^4 - 4q^2 + 5 - 4q^{-2} + 3q^{-4} - q^{-6} \quad (209)$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = -q^{12} + 3q^8 - 4q^4 + 5 - 4q^{-4} + 3q^{-8} - q^{-12} \quad (210)$$

Jones polynomials

$$J_{[1]} = 1 - q^{-2} + 3q^{-4} - 3q^{-6} + 3q^{-8} - 4q^{-10} + 3q^{-12} - 2q^{-14} + q^{-16} \quad (211)$$

$$J_{[1,1]} = 1 \quad (212)$$

$$\begin{aligned} J_{[2]} &= q^4 - q^2 - 1 + 4q^{-2} - q^{-4} - 5q^{-6} + 7q^{-8} - 9q^{-12} + 8q^{-14} + 3q^{-16} - 12q^{-18} + 7q^{-20} + 6q^{-22} - 12q^{-24} + \\ &\quad + 5q^{-26} + 7q^{-28} - 10q^{-30} + 3q^{-32} + 5q^{-34} - 6q^{-36} + 2q^{-38} + q^{-40} - 2q^{-42} + q^{-44} \end{aligned} \quad (213)$$

Special polynomials

$$\mathfrak{H}_{[1]} = 2A^{-6} - 5A^{-4} + 4A^{-2} \quad (214)$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 4A^{-12} - 20A^{-10} + 41A^{-8} - 40A^{-6} + 16A^{-4} \quad (215)$$

Ooguri-Vafa polynomials

$$\begin{aligned}
f_{[1,1]} &= \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left((q^{19} - q^{17} + 2q^{15} + 2q^9 + q^7 + q^3 + 3q^1 - 2q^{-1} + 2q^{-3})A^{-10} + \right. \\
&\quad \left. + (-q^{15} - 2q^{11} - q^9 - 2q^7 - q^5 - 5q^3 + 2q^1 - 4q^{-1} - 2q^{-5} + q^{-7} - q^{-9})A^{-8} + (q^9 + 2q^5 + q^1 - 2q^{-5} - q^{-11})A^{-6} \right) \\
f_{[2]} &= \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left((-2q^3 + 2q^1 - 3q^{-1} - q^{-3} - q^{-7} - 2q^{-9} - 2q^{-15} + q^{-17} - q^{-19})A^{-10} + \right. \\
&\quad \left. + (q^9 - q^7 + 2q^5 + 4q^1 - 2q^{-1} + 5q^{-3} + q^{-5} + 2q^{-7} + q^{-9} + 2q^{-11} + q^{-15})A^{-8} + (q^{11} + 2q^5 - q^{-1} - 2q^{-5} - q^{-9})A^{-6} \right)
\end{aligned} \tag{216}$$

Special Ooguri-Vafa polynomials

$$f_{[2]} = -f_{[1,1]} = -\frac{(A^4 - 16A^2 + 9)(A - A^{-1})^3}{A^{10}} \tag{218}$$

Numbers $N_{R,n,k}$

$k \backslash n =$	-7	-5	-3	-1
0	-2	7	-9	4
1	-3	11	-12	4
2	-1	6	-6	1
3	0	1	-1	0

$N_{[1]} :$

$k \backslash n =$	-14	-12	-10	-8	-6	-4	-2
0	25	-123	262	-318	237	-103	20
1	170	-802	1536	-1573	951	-337	55
2	475	-2229	3987	-3529	1702	-470	64
3	704	-3421	5890	-4601	1733	-342	37
4	605	-3180	5402	-3797	1093	-133	10
5	310	-1857	3175	-2043	440	-26	1
6	93	-683	1194	-713	111	-2	0
7	15	-153	277	-155	16	0	0
8	1	-19	36	-19	1	0	0
9	0	-1	2	-1	0	0	0

$N_{[1,1]} :$

$k \backslash n =$	-14	-12	-10	-8	-6	-4	-2
0	34	-175	381	-454	316	-123	21
1	264	-1285	2521	-2559	1425	-416	50
2	882	-4185	7631	-6766	3013	-619	44
3	1601	-7697	13423	-10646	3822	-519	16
4	1729	-8758	14953	-10809	3148	-265	2
5	1157	-6437	10961	-7321	1722	-82	0
6	483	-3103	5348	-3333	619	-14	0
7	122	-972	1716	-1005	140	-1	0
8	17	-190	347	-192	18	0	0
9	1	-21	40	-21	1	0	0
10	0	-1	2	-1	0	0	0

$N_{[2]} :$

Knot 8₇

$$(-2, 1 | -1, 4)$$

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= A^{-2} \left((-q^4 + q^2 - 2 + q^{-2} - q^{-4})A^{-2} + \right. \\
&\quad \left. + (q^6 - q^4 + 3q^2 - 2 + 3q^{-2} - q^{-4} + q^{-6}) + (-q^4 + q^2 - 1 + q^{-2} - q^{-4})A^2 \right) = \\
&= \frac{A^{-2}}{*S_{[1]}} \left(*S_{[3]}q^2 + *S_{[2,1]}(-q^8 + 2q^6 - 3q^4 + 4q^2 - 5 + 4q^{-2} - 3q^{-4} + 2q^{-6} - q^{-8}) + *S_{[1,1,1]}q^{-2} \right) \\
\frac{H_{[1,1]}}{*S_{[1,1]}} &= A^{-4}q^8 \left((q^{10} - q^8 + 2q^4 - 2q^2 + 4q^{-2} - 2q^{-4} - q^{-6} + 3q^{-8} - q^{-12} + q^{-14})A^{-4} + \right. \\
&\quad \left. + (-q^{12} + q^8 - 3q^6 + 2q^2 - 7 + 5q^{-4} - 8q^{-6} - 4q^{-8} + \right. \\
&\quad \left. + 5q^{-10} - 2q^{-12} - 4q^{-14} + q^{-16} - q^{-20})A^{-2} + \right. \\
&\quad \left. + (q^{12} - q^{10} + q^8 + 3q^6 - 2q^4 + 2q^2 + 5 - 5q^{-2} + 5q^{-4} + 8q^{-6} - \right. \\
&\quad \left. - 7q^{-8} + 10q^{-12} - 2q^{-14} - 3q^{-16} + 4q^{-18} + q^{-20} - q^{-22} + q^{-24}) + \right. \\
&\quad \left. + (-q^8 + q^4 - 3q^2 + q^{-2} - 4q^{-4} + 2q^{-6} + 2q^{-8} - 6q^{-10} + 5q^{-14} - 3q^{-16} - 3q^{-18} + 2q^{-20} - q^{-24})A^2 + \right. \\
&\quad \left. + (q^2 - 1 + q^{-4} - q^{-6} + q^{-8} + q^{-10} - 2q^{-12} + 2q^{-16} - q^{-18} - q^{-20} + q^{-22})A^4 \right) = \\
&= \frac{A^{-4}q^8}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(-q^6 + 2q^4 - 3q^2 + 4 - 5q^{-2} + 4q^{-4} - 3q^{-6} + 2q^{-8} - q^{-10}) + *S_{[3,1,1,1]}q^{-4} + \right. \\
&\quad \left. + *S_{[2,2,2]}q^{-4} + *S_{[2,2,1,1]}(q^{18} - 2q^{16} + 4q^{12} - 5q^{10} + q^8 + 7q^6 - 11q^4 + 4q^2 + 11 - 19q^{-2} + 5q^{-4} + \right. \\
&\quad \left. + 17q^{-6} - 20q^{-8} - q^{-10} + 19q^{-12} - 12q^{-14} - 7q^{-16} + 13q^{-18} - 3q^{-20} - 6q^{-22} + 5q^{-24} - 2q^{-28} + q^{-30}) + \right. \\
&\quad \left. + *S_{[2,1,1,1,1]}(-q^8 + 2q^4 - 3 + 4q^{-4} - 5q^{-8} + 4q^{-12} - 3q^{-16} + 2q^{-20} - q^{-24}) + *S_{[1,1,1,1,1,1]}q^{-12} \right) \\
\frac{H_{[2]}}{*S_{[2]}} &= A^{-4}q^{-8} \left((q^{14} - q^{12} + 3q^8 - q^6 - 2q^4 + 4q^2 - 2q^{-2} + 2q^{-4} - q^{-8} + q^{-10})A^{-4} + \right. \\
&\quad \left. + (-q^{20} + q^{16} - 4q^{14} - 2q^{12} + 5q^{10} - 4q^8 - 8q^6 + 5q^4 - 7 + 2q^{-2} - 3q^{-6} + q^{-8} - q^{-12})A^{-2} + \right. \\
&\quad \left. + (q^{24} - q^{22} + q^{20} + 4q^{18} - 3q^{16} - 2q^{14} + 10q^{12} - 7q^8 + 8q^6 + 5q^4 - 5q^2 + 5 + 2q^{-2} - 2q^{-4} + 3q^{-6} + q^{-8} - q^{-10} + q^{-12}) + \right. \\
&\quad \left. + (-q^{24} + 2q^{20} - 3q^{18} - 3q^{16} + 5q^{14} - 6q^{10} + 2q^8 + 2q^6 - 4q^4 + q^2 - 3q^{-2} + q^{-4} - q^{-8})A^2 + \right. \\
&\quad \left. + (q^{22} - q^{20} - q^{18} + 2q^{16} - 2q^{12} + q^{10} + q^8 - q^6 + q^4 - 1 + q^{-2})A^4 \right) = \\
&= \frac{A^{-4}q^{-8}}{*S_{[2]}} \left(*S_{[6]}q^{12} + *S_{[5,1]}(-q^{24} + 2q^{20} - 3q^{16} + 4q^{12} - 5q^8 + 4q^4 - 3 + 2q^{-4} - q^{-8}) + \right. \\
&\quad \left. + *S_{[4,2]}(q^{30} - 2q^{28} + 5q^{24} - 6q^{22} - 3q^{20} + 13q^{18} - 7q^{16} - 12q^{14} + 19q^{12} - q^{10} - 20q^8 + \right. \\
&\quad \left. + 17q^6 + 5q^4 - 19q^2 + 11 + 4q^{-2} - 11q^{-4} + 7q^{-6} + q^{-8} - 5q^{-10} + 4q^{-12} - 2q^{-16} + q^{-18}) + \right. \\
&\quad \left. + *S_{[4,1,1]}q^4 + *S_{[3,3]}q^4 + *S_{[3,2,1]}(-q^{10} + 2q^8 - 3q^6 + 4q^4 - 5q^2 + 4 - 3q^{-2} + 2q^{-4} - q^{-6}) + *S_{[2,2,2]} \right)
\end{aligned} \tag{221}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = q^6 - 3q^4 + 5q^2 - 5 + 5q^{-2} - 3q^{-4} + q^{-6} \tag{222}$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = q^{12} - 3q^8 + 5q^4 - 5 + 5q^{-4} - 3q^{-8} + q^{-12} \tag{223}$$

Jones polynomials

$$J_{[1]} = -q^4 + 2q^2 - 2 + 4q^{-2} - 4q^{-4} + 4q^{-6} - 3q^{-8} + 2q^{-10} - q^{-12} \tag{224}$$

$$J_{[1,1]} = 1 \tag{225}$$

$$\begin{aligned}
J_{[2]} &= q^{14} - 2q^{12} - q^{10} + 5q^8 - 4q^6 - 4q^4 + 10q^2 - 3 - 9q^{-2} + 14q^{-4} - 2q^{-6} - 14q^{-8} + \\
&\quad + 16q^{-10} - 16q^{-14} + 14q^{-16} + q^{-18} - 12q^{-20} + 9q^{-22} + q^{-24} - 6q^{-26} + 4q^{-28} - 2q^{-32} + q^{-34}
\end{aligned} \tag{226}$$

Special polynomials

$$\mathfrak{H}_{[1]} = -2A^{-4} + 4A^{-2} - 1 \quad (227)$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 4A^{-8} - 16A^{-6} + 20A^{-4} - 8A^{-2} + 1 \quad (228)$$

Ooguri-Vafa polynomials

$$f_{[1,1]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2})}{\{q\}} \left((q^{17} + q^{11} + 3q^5 + q^{-3})A^{-6} + \right. \\ \left. + (-q^{13} - q^{11} - 2q^7 + q^5 - 2q^{-1} + 2q^{-3} + q^{-5})A^{-4} + (q^1 - q^{-1} + q^{-5} - q^{-9})A^{-2} \right) \quad (229)$$

$$f_{[2]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2})}{\{q\}} \left((-q^7 - 3q^{-1} - q^{-7} - q^{-13})A^{-6} + \right. \\ \left. + (-q^9 - 2q^7 + 2q^5 - q^{-1} + 2q^{-3} + q^{-7} + q^{-9})A^{-4} + (q^{13} - q^9 + q^5 - q^3)A^{-2} \right) \quad (230)$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = -\frac{2(A^3 - 3)(A - A^{-1})^3}{A^6} \quad (231)$$

Numbers $N_{R,n,k}$

						$k \backslash n =$	-2	0	2	4	6	8	10
$N_{[1]} :$	$k \backslash n =$					0	39	-286	785	-1028	665	-194	19
						1	275	-2553	8365	-12718	9477	-3227	381
						2	853	-10436	41492	-73336	62457	-24578	3548
						3	1497	-25234	123381	-252783	242378	-107162	17923
						4	1617	-39707	242296	-573983	610752	-294896	53921
						5	1103	-42639	331047	-906126	1057291	-544430	103754
$N_{[1,1]} :$	0	2	-6	5	-1	6	471	-31998	324752	-1028553	1303219	-701974	134083
	1	3	-11	11	-3	7	121	-16898	232948	-857227	1169635	-648615	120036
	2	1	-6	6	-1	8	17	-6230	123125	-530576	773886	-435958	75736
	3	0	-1	1	0	9	1	-1564	47840	-244654	378959	-214440	33858
						10	0	-254	13479	-83606	136710	-76977	10648
						11	0	-24	2675	-20851	35800	-19901	2301
						12	0	-1	354	-3683	6608	-3603	325
						13	0	0	28	-436	814	-433	27
						14	0	0	1	-31	60	-31	1
						15	0	0	0	-1	2	-1	0

$k \setminus n =$	-2	0	2	4	6	8	10
0	29	-216	605	-808	531	-156	15
1	173	-1685	5733	-8988	6875	-2397	289
2	451	-6003	25170	-46202	40551	-16425	2458
3	654	-12559	65901	-141221	139859	-63619	10985
4	570	-16938	113269	-282887	311420	-154255	28821
5	300	-15385	134426	-391395	473113	-248899	47840
6	92	-9578	113396	-386023	507192	-277756	52677
7	15	-4078	68991	-276387	391286	-219388	39561
8	1	-1162	30345	-144784	219153	-124072	20519
9	0	-211	9549	-55385	89010	-50298	7335
10	0	-22	2093	-15273	25898	-14468	1772
11	0	-1	303	-2952	5252	-2878	276
12	0	0	26	-379	704	-376	25
13	0	0	1	-29	56	-29	1
14	0	0	0	-1	2	-1	0

$N_{[2]} :$

Knot 8_9

$(3, -1|1, -3)$

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= (q^4 - q^2 + 2 - q^{-2} + q^{-4})A^{-2} + (-q^6 + q^4 - 3q^2 + 3 - 3q^{-2} + q^{-4} - q^{-6}) + (q^4 - q^2 + 2 - q^{-2} + q^{-4})A^2 = \\
&= \frac{1}{*S_{[1]}} \left(*S_{[3]} + *S_{[2,1]}(q^8 - 2q^6 + 3q^4 - 5q^2 + 5 - 5q^{-2} + 3q^{-4} - 2q^{-6} + q^{-8}) + *S_{[1,1,1]} \right)
\end{aligned} \tag{232}$$

$$\begin{aligned}
& \frac{H_{[1,1]}}{*S_{[1,1]}} = (q^{16} - q^{14} + 3q^{10} - q^8 - 2q^6 + 4q^4 - 2 + 2q^{-2} - q^{-6} + q^{-8})A^{-4} + \\
& + (-q^{18} + q^{14} - 4q^{12} - q^{10} + 5q^8 - 6q^6 - 6q^4 + 8q^2 - 2 - 7q^{-2} + 4q^{-4} - 3q^{-8} + q^{-10} - q^{-14})A^{-2} + \\
& + (q^{18} - q^{16} + q^{14} + 3q^{12} - 3q^{10} + q^8 + 9q^6 - 7q^4 - 3q^2 + 15 - \\
& - 3q^{-2} - 7q^{-4} + 9q^{-6} + q^{-8} - 3q^{-10} + 3q^{-12} + q^{-14} - q^{-16} + q^{-18}) + \\
& + (-q^{14} + q^{10} - 3q^8 + 4q^4 - 7q^2 - 2 + 8q^{-2} - 6q^{-4} - 6q^{-6} + 5q^{-8} - q^{-10} - 4q^{-12} + q^{-14} - q^{-18})A^2 + \\
& + (q^8 - q^6 + 2q^2 - 2 + 4q^{-4} - 2q^{-6} - q^{-8} + 3q^{-10} - q^{-14} + q^{-16})A^4 = \\
& = \frac{1}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(q^8 - 2q^6 + 3q^4 - 5q^2 + 5 - 5q^{-2} + 3q^{-4} - 2q^{-6} + q^{-8}) + *S_{[3,1,1,1]} + *S_{[2,2,2]} + \right. \\
& + *S_{[2,2,1,1]}(q^{24} - 2q^{22} + 4q^{18} - 6q^{16} + 11q^{12} - 13q^{10} - 4q^8 + 24q^6 - 18q^4 - 13q^2 + \\
& + 32 - 13q^{-2} - 18q^{-4} + 24q^{-6} - 4q^{-8} - 13q^{-10} + 11q^{-12} - 6q^{-16} + 4q^{-18} - 2q^{-22} + q^{-24}) + \\
& + *S_{[2,1,1,1,1]}(q^{16} - 2q^{12} + 3q^8 - 5q^4 + 5 - 5q^{-4} + 3q^{-8} - 2q^{-12} + q^{-16}) + *S_{[1,1,1,1,1,1]} \Big) \\
& \frac{H_{[2]}}{*S_{[2]}} = (q^8 - q^6 + 2q^2 - 2 + 4q^{-4} - 2q^{-6} - q^{-8} + 3q^{-10} - q^{-14} + q^{-16})A^{-4} + \\
& + (-q^{14} + q^{10} - 3q^8 + 4q^4 - 7q^2 - 2 + 8q^{-2} - 6q^{-4} - 6q^{-6} + 5q^{-8} - q^{-10} - 4q^{-12} + q^{-14} - q^{-18})A^{-2} + \\
& + (q^{18} - q^{16} + q^{14} + 3q^{12} - 3q^{10} + q^8 + 9q^6 - 7q^4 - 3q^2 + 15 - \\
& - 3q^{-2} - 7q^{-4} + 9q^{-6} + q^{-8} - 3q^{-10} + 3q^{-12} + q^{-14} - q^{-16} + q^{-18}) + \\
& + (-q^{18} + q^{14} - 4q^{12} - q^{10} + 5q^8 - 6q^6 - 6q^4 + 8q^2 - 2 - 7q^{-2} + 4q^{-4} - 3q^{-8} + q^{-10} - q^{-14})A^2 + \\
& + (q^{16} - q^{14} + 3q^{10} - q^8 - 2q^6 + 4q^4 - 2 + 2q^{-2} - q^{-6} + q^{-8})A^4 = \\
& = \frac{1}{*S_{[2]}} \left(*S_{[6]} + *S_{[5,1]}(q^{16} - 2q^{12} + 3q^8 - 5q^4 + 5 - 5q^{-4} + 3q^{-8} - 2q^{-12} + q^{-16}) + \right. \\
& + *S_{[4,2]}(q^{24} - 2q^{22} + 4q^{18} - 6q^{16} + 11q^{12} - 13q^{10} - 4q^8 + 24q^6 - 18q^4 - 13q^2 + \\
& + 32 - 13q^{-2} - 18q^{-4} + 24q^{-6} - 4q^{-8} - 13q^{-10} + 11q^{-12} - 6q^{-16} + 4q^{-18} - 2q^{-22} + q^{-24}) + \\
& + *S_{[4,1,1]} + *S_{[3,3]} + *S_{[3,2,1]}(q^8 - 2q^6 + 3q^4 - 5q^2 + 5 - 5q^{-2} + 3q^{-4} - 2q^{-6} + q^{-8}) + *S_{[2,2,2]} \Big)
\end{aligned} \tag{233}$$

$$\begin{aligned}
& \frac{H_{[2]}}{*S_{[2]}} = (q^8 - q^6 + 2q^2 - 2 + 4q^{-4} - 2q^{-6} - q^{-8} + 3q^{-10} - q^{-14} + q^{-16})A^{-4} + \\
& + (-q^{14} + q^{10} - 3q^8 + 4q^4 - 7q^2 - 2 + 8q^{-2} - 6q^{-4} - 6q^{-6} + 5q^{-8} - q^{-10} - 4q^{-12} + q^{-14} - q^{-18})A^{-2} + \\
& + (q^{18} - q^{16} + q^{14} + 3q^{12} - 3q^{10} + q^8 + 9q^6 - 7q^4 - 3q^2 + 15 - \\
& - 3q^{-2} - 7q^{-4} + 9q^{-6} + q^{-8} - 3q^{-10} + 3q^{-12} + q^{-14} - q^{-16} + q^{-18}) + \\
& + (-q^{18} + q^{14} - 4q^{12} - q^{10} + 5q^8 - 6q^6 - 6q^4 + 8q^2 - 2 - 7q^{-2} + 4q^{-4} - 3q^{-8} + q^{-10} - q^{-14})A^2 + \\
& + (q^{16} - q^{14} + 3q^{10} - q^8 - 2q^6 + 4q^4 - 2 + 2q^{-2} - q^{-6} + q^{-8})A^4 = \\
& = \frac{1}{*S_{[2]}} \left(*S_{[6]} + *S_{[5,1]}(q^{16} - 2q^{12} + 3q^8 - 5q^4 + 5 - 5q^{-4} + 3q^{-8} - 2q^{-12} + q^{-16}) + \right. \\
& + *S_{[4,2]}(q^{24} - 2q^{22} + 4q^{18} - 6q^{16} + 11q^{12} - 13q^{10} - 4q^8 + 24q^6 - 18q^4 - 13q^2 + \\
& + 32 - 13q^{-2} - 18q^{-4} + 24q^{-6} - 4q^{-8} - 13q^{-10} + 11q^{-12} - 6q^{-16} + 4q^{-18} - 2q^{-22} + q^{-24}) + \\
& + *S_{[4,1,1]} + *S_{[3,3]} + *S_{[3,2,1]}(q^8 - 2q^6 + 3q^4 - 5q^2 + 5 - 5q^{-2} + 3q^{-4} - 2q^{-6} + q^{-8}) + *S_{[2,2,2]} \Big)
\end{aligned} \tag{234}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = -q^6 + 3q^4 - 5q^2 + 7 - 5q^{-2} + 3q^{-4} - q^{-6} \tag{235}$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = -q^{12} + 3q^8 - 5q^4 + 7 - 5q^{-4} + 3q^{-8} - q^{-12} \tag{236}$$

Jones polynomials

$$J_{[1]} = q^8 - 2q^6 + 3q^4 - 4q^2 + 5 - 4q^{-2} + 3q^{-4} - 2q^{-6} + q^{-8} \tag{237}$$

$$J_{[1,1]} = 1 \tag{238}$$

$$\begin{aligned}
J_{[2]} &= q^{24} - 2q^{22} + 5q^{18} - 6q^{16} - 2q^{14} + 12q^{12} - 10q^{10} - 7q^8 + 20q^6 - 12q^4 - 11q^2 + \\
&+ 25 - 11q^{-2} - 12q^{-4} + 20q^{-6} - 7q^{-8} - 10q^{-10} + 12q^{-12} - 2q^{-14} - 6q^{-16} + 5q^{-18} - 2q^{-22} + q^{-24}
\end{aligned} \tag{239}$$

Special polynomials

$$\mathfrak{H}_{[1]} = 2A^{-2} - 3 + 2A^2 \tag{240}$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 4A^{-4} - 12A^{-2} + 17 - 12A^2 + 4A^4 \tag{241}$$

Ooguri-Vafa polynomials

$$\begin{aligned}
f_{[1,1]} &= \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left((q^{15} + 2q^9 + q^7 + q^3 + 2q^1 - q^{-1})A^{-2} + \right. \\
&+ (-q^{11} + q^7 + q^5 - 3q^3 + 3q^1 - q^{-1} - q^{-3} + q^{-7}) + (q^5 - 2q^3 - q^1 - q^{-3} - 2q^{-5} - q^{-11})A^2 \Big)
\end{aligned} \tag{242}$$

$$\begin{aligned}
f_{[2]} &= \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left((q^5 - 2q^3 - q^1 - q^{-3} - 2q^{-5} - q^{-11})A^{-2} + \right. \\
&+ (-q^{11} + q^7 + q^5 - 3q^3 + 3q^1 - q^{-1} - q^{-3} + q^{-7}) + (q^{15} + 2q^9 + q^7 + q^3 + 2q^1 - q^{-1})A^2 \Big)
\end{aligned} \tag{243}$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = 6(A^2 - A^{-2})(A - A^{-1})^3$$

(244)

Numbers $N_{R,n,k}$

$N_{[1]} :$						$N_{[1,1]} :$										
		$k \backslash n =$	-3	-1	1		3		$k \backslash n =$	-6	-4	-2	0	2	4	6
	0	-2	5	-5	2		0	14	-60	120	-160	150	-84	20		
	1	-3	11	-11	3		1	59	-278	598	-846	815	-448	100		
	2	-1	6	-6	1		2	103	-521	1183	-1803	1845	-1026	219		
	3	0	1	-1	0		3	94	-514	1221	-2052	2288	-1298	261		
							4	46	-288	716	-1377	1705	-979	177		
							5	11	-91	239	-561	781	-446	67		
							6	1	-15	42	-136	214	-119	13		
					7	0	-1	3	-18	32	-17	1				
					8	0	0	0	-1	2	-1	0				

		$k \backslash n =$	-6	-4	-2	0	2	4	6
$N_{[2]} :$	0	20	-84	150	-160	120	-60	14	
	1	100	-448	815	-846	598	-278	59	
	2	219	-1026	1845	-1803	1183	-521	103	
	3	261	-1298	2288	-2052	1221	-514	94	
	4	177	-979	1705	-1377	716	-288	46	
	5	67	-446	781	-561	239	-91	11	
	6	13	-119	214	-136	42	-15	1	
	7	1	-17	32	-18	3	-1	0	
	8	0	-1	2	-1	0	0	0	

Knot 8₁₀

$(-2, 2 | -1, 3)$

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= A^{-2} \left((-q^4 + q^2 - 3 + q^{-2} - q^{-4})A^{-2} + \right. \\
&\quad \left. + (q^6 - q^4 + 4q^2 - 2 + 4q^{-2} - q^{-4} + q^{-6}) + (-q^4 + q^2 - 2 + q^{-2} - q^{-4})A^2 \right) = \\
&= \frac{A^{-2}}{*S_{[1]}} \left(*S_{[3]}q^2 + *S_{[2,1]}(-q^8 + 2q^6 - 4q^4 + 5q^2 - 5 + 5q^{-2} - 4q^{-4} + 2q^{-6} - q^{-8}) + *S_{[1,1,1]}q^{-2} \right) \\
\frac{H_{[1,1]}}{*S_{[1,1]}} &= A^{-4}q^8 \left((q^{10} - q^8 + q^6 + 3q^4 - 3q^2 + 1 + 7q^{-2} - 3q^{-4} - q^{-6} + 4q^{-8} - q^{-12} + q^{-14})A^{-4} + \right. \\
&\quad \left. + (-q^{12} - 5q^6 + q^2 - 13 - 2q^{-2} + 6q^{-4} - 13q^{-6} - 7q^{-8} + 6q^{-10} - 3q^{-12} - 5q^{-14} + q^{-16} - q^{-20})A^{-2} + \right. \\
&\quad \left. + (q^{12} - q^{10} + 2q^8 + 4q^6 - 2q^4 + 6q^2 + 10 - 6q^{-2} + 10q^{-4} + 15q^{-6} - \right. \\
&\quad \left. - 8q^{-8} + q^{-10} + 15q^{-12} - 2q^{-14} - 3q^{-16} + 5q^{-18} + q^{-20} - q^{-22} + q^{-24}) + \right. \\
&\quad \left. + (-q^8 - 5q^2 - 9q^{-4} + q^{-6} + 3q^{-8} - 10q^{-10} - 2q^{-12} + 6q^{-14} - 4q^{-16} - 4q^{-18} + 2q^{-20} - q^{-24})A^2 + \right. \\
&\quad \left. + (q^2 - 1 + q^{-2} + 2q^{-4} - 2q^{-6} + q^{-8} + 3q^{-10} - 3q^{-12} + 3q^{-16} - q^{-18} - q^{-20} + q^{-22})A^4 \right) = \\
&= \frac{A^{-4}q^8}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(-q^6 + 2q^4 - 4q^2 + 5 - 5q^{-2} + 5q^{-4} - 4q^{-6} + 2q^{-8} - q^{-10}) + *S_{[3,1,1,1]}q^{-4} + \right. \\
&\quad + *S_{[2,2,2]}q^{-4} + *S_{[2,2,1,1]}(q^{18} - 2q^{16} + q^{14} + 4q^{12} - 8q^{10} + 3q^8 + 10q^6 - 18q^4 + 7q^2 + 17 - 28q^{-2} + 8q^{-4} + \\
&\quad + 24q^{-6} - 29q^{-8} - q^{-10} + 27q^{-12} - 18q^{-14} - 9q^{-16} + 18q^{-18} - 5q^{-20} - 7q^{-22} + 6q^{-24} - 2q^{-28} + q^{-30}) + \\
&\quad + *S_{[2,1,1,1,1]}(-q^8 + 2q^4 - 4 + 5q^{-4} - 5q^{-8} + 5q^{-12} - 4q^{-16} + 2q^{-20} - q^{-24}) + *S_{[1,1,1,1,1,1]}q^{-12} \Big) \\
\frac{H_{[2]}}{*S_{[2]}} &= A^{-4}q^{-8} \left((q^{14} - q^{12} + 4q^8 - q^6 - 3q^4 + 7q^2 + 1 - 3q^{-2} + 3q^{-4} + q^{-6} - q^{-8} + q^{-10})A^{-4} + \right. \\
&\quad \left. + (-q^{20} + q^{16} - 5q^{14} - 3q^{12} + 6q^{10} - 7q^8 - 13q^6 + 6q^4 - 2q^2 - 13 + q^{-2} - 5q^{-6} - q^{-12})A^{-2} + \right. \\
&\quad \left. + (q^{24} - q^{22} + q^{20} + 5q^{18} - 3q^{16} - 2q^{14} + 15q^{12} + q^{10} - 8q^8 + \right. \\
&\quad \left. + 15q^6 + 10q^4 - 6q^2 + 10 + 6q^{-2} - 2q^{-4} + 4q^{-6} + 2q^{-8} - q^{-10} + q^{-12}) + \right. \\
&\quad \left. + (-q^{24} + 2q^{20} - 4q^{18} - 4q^{16} + 6q^{14} - 2q^{12} - 10q^{10} + 3q^8 + q^6 - 9q^4 - 5q^{-2} - q^{-8})A^2 + \right. \\
&\quad \left. + (q^{22} - q^{20} - q^{18} + 3q^{16} - 3q^{12} + 3q^{10} + q^8 - 2q^6 + 2q^4 + q^2 - 1 + q^{-2})A^4 \right) = \\
&= \frac{A^{-4}q^{-8}}{*S_{[2]}} \left(*S_{[6]}q^{12} + *S_{[5,1]}(-q^{24} + 2q^{20} - 4q^{16} + 5q^{12} - 5q^8 + 5q^4 - 4 + 2q^{-4} - q^{-8}) + \right. \\
&\quad + *S_{[4,2]}(q^{30} - 2q^{28} + 6q^{24} - 7q^{22} - 5q^{20} + 18q^{18} - 9q^{16} - 18q^{14} + 27q^{12} - q^{10} - 29q^8 + \\
&\quad + 24q^6 + 8q^4 - 28q^2 + 17 + 7q^{-2} - 18q^{-4} + 10q^{-6} + 3q^{-8} - 8q^{-10} + 4q^{-12} + q^{-14} - 2q^{-16} + q^{-18}) + \\
&\quad \left. + *S_{[4,1,1]}q^4 + *S_{[3,3]}q^4 + *S_{[3,2,1]}(-q^{10} + 2q^8 - 4q^6 + 5q^4 - 5q^2 + 5 - 4q^{-2} + 2q^{-4} - q^{-6}) + *S_{[2,2,2]} \right)
\end{aligned}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = q^6 - 3q^4 + 6q^2 - 7 + 6q^{-2} - 3q^{-4} + q^{-6} \quad (248)$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = q^{12} - 3q^8 + 6q^4 - 7 + 6q^{-4} - 3q^{-8} + q^{-12} \quad (249)$$

Jones polynomials

$$J_{[1]} = -q^4 + 2q^2 - 3 + 5q^{-2} - 4q^{-4} + 5q^{-6} - 4q^{-8} + 2q^{-10} - q^{-12} \quad (250)$$

$$J_{[1,1]} = 1 \quad (251)$$

$$J_{[2]} = q^{14} - 2q^{12} - q^{10} + 6q^8 - 5q^6 - 6q^4 + 14q^2 - 5 - 14q^{-2} + 21q^{-4} - 2q^{-6} - 21q^{-8} + 23q^{-10} + \quad (252)$$

$$+ 2q^{-12} - 24q^{-14} + 20q^{-16} + 3q^{-18} - 19q^{-20} + 12q^{-22} + 3q^{-24} - 9q^{-26} + 4q^{-28} + q^{-30} - 2q^{-32} + q^{-34} \quad (253)$$

Special polynomials

$$\mathfrak{H}_{[1]} = -3A^{-4} + 6A^{-2} - 2 \quad (254)$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 9A^{-8} - 36A^{-6} + 48A^{-4} - 24A^{-2} + 4 \quad (255)$$

Ooguri-Vafa polynomials

$$f_{[1,1]} = \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left((q^{17} - q^{15} + 2q^{13} + 2q^{11} - q^9 + 3q^7 + 6q^5 - 3q^3 + 6q^1 + 2q^{-3} - q^{-5} + q^{-7})A^{-6} + \right. \\ \left. + (-q^{13} - q^9 - 5q^7 + 3q^5 - 5q^3 - 4q^{-1} + 5q^{-3} - 4q^{-5} + q^{-7} + q^{-9})A^{-4} + \right. \\ \left. + (2q^1 - 4q^{-1} + 3q^{-3} - 2q^{-7} + q^{-11} - q^{-13})A^{-2} \right) \quad (256)$$

$$f_{[2]} = \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left((-q^7 + q^5 - 2q^3 - 6q^{-1} + 3q^{-3} - 6q^{-5} - 3q^{-7} + q^{-9} - 2q^{-11} - 2q^{-13} + q^{-15} - q^{-17})A^{-6} + \right. \\ \left. + (-q^9 - q^7 + 4q^5 - 5q^3 + 4q^1 + 5q^{-3} - 3q^{-5} + 5q^{-7} + q^{-9} + q^{-13})A^{-4} + (q^{13} - q^{11} + 2q^7 - 3q^3 + 4q^1 - 2q^{-1})A^{-2} \right) \quad (257)$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = \frac{6(A^4 + 10A^2 - 17)(A - A^{-1})^3}{A^6} \quad (258)$$

Numbers $N_{R,n,k}$

		$k \setminus n =$				$k \setminus n =$			
		-5	-3	-1	1	-10	-8	-6	-4
$N_{[1]} :$	0	3	-9	8	-2	32	-155	307	-318
	1	3	-12	12	-3	138	-665	1281	-1283
	2	1	-6	6	-1	272	-1316	2431	-2294
	3	0	-1	1	0	296	-1502	2680	-2338
						187	-1055	1849	-1468
		$N_{[1,1]} :$							
						719	-277	45	
						68	-460	808	-576
						239	-90	11	
						13	-120	216	-137
						42	-15	1	
						1	-17	32	-18
						3	-1	0	
						0	-1	2	-1
						0	0	0	0

$k \setminus n =$	-10	-8	-6	-4	-2	0	2
0	49	-233	448	-442	233	-61	6
1	246	-1162	2150	-1979	949	-227	23
2	576	-2731	4838	-4065	1697	-352	37
3	769	-3786	6511	-4967	1728	-283	28
4	624	-3321	5649	-3933	1092	-120	9
5	312	-1884	3225	-2069	440	-25	1
6	93	-685	1198	-715	111	-2	0
7	15	-153	277	-155	16	0	0
8	1	-19	36	-19	1	0	0
9	0	-1	2	-1	0	0	0

$N_{[2]} :$

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$(1, -1|1, -2|1, -2)$

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= A^2 \left((-q^4 + 2q^2 - 2 + 2q^{-2} - q^{-4})A^{-2} + \right. \\
&\quad \left. + (q^6 - 2q^4 + 4q^2 - 4 + 4q^{-2} - 2q^{-4} + q^{-6}) + (-q^4 + 2q^2 - 3 + 2q^{-2} - q^{-4})A^2 \right) = \\
&= \frac{A^2}{*S_{[1]}} \left(*S_{[3]}q^{-2} + *S_{[2,1]}(-q^8 + 3q^6 - 5q^4 + 6q^2 - 7 + 6q^{-2} - 5q^{-4} + 3q^{-6} - q^{-8}) + *S_{[1,1,1]}q^2 \right) \\
\frac{H_{[1,1]}}{*S_{[1,1]}} &= A^4 q^{-8} \left((q^{22} - 2q^{20} - 2q^{18} + 5q^{16} - q^{14} - 6q^{12} + 5q^{10} + 2q^8 - 5q^6 + 3q^4 + q^2 - 2 + q^{-2})A^{-4} + \right. \\
&\quad + (-q^{24} + q^{22} + 4q^{20} - 5q^{18} - 4q^{16} + 13q^{14} - q^{12} - 14q^{10} + \\
&\quad + 10q^8 + 6q^6 - 12q^4 + 4q^2 + 4 - 6q^{-2} + q^{-4} + q^{-6} - q^{-8})A^{-2} + \\
&\quad + (q^{24} - 2q^{22} + 6q^{18} - 8q^{16} - 7q^{14} + 19q^{12} - 4q^{10} - 21q^8 + \\
&\quad + 18q^6 + 8q^4 - 18q^2 + 9 + 6q^{-2} - 8q^{-4} + 4q^{-6} + 2q^{-8} - 2q^{-10} + q^{-12}) + \\
&\quad + (-q^{20} + q^{18} + 3q^{16} - 6q^{14} - 2q^{12} + 14q^{10} - 6q^8 - 16q^6 + \\
&\quad + 16q^4 + 4q^2 - 16 + 6q^{-2} + 4q^{-4} - 6q^{-6} + q^{-8} + q^{-10} - q^{-12})A^2 + \\
&\quad \left. + (q^{14} - 2q^{12} - q^{10} + 6q^8 - 3q^6 - 7q^4 + 9q^2 + 1 - 7q^{-2} + 4q^{-4} + q^{-6} - 2q^{-8} + q^{-10})A^4 \right) = \\
&= \frac{A^4 q^{-8}}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(-q^{10} + 3q^8 - 5q^6 + 6q^4 - 7q^2 + 6 - 5q^{-2} + 3q^{-4} - q^{-6}) + *S_{[3,1,1,1]}q^4 + *S_{[2,2,2]}q^4 + \right. \\
&\quad + *S_{[2,2,1,1]}(q^{30} - 3q^{28} + 10q^{24} - 11q^{22} - 9q^{20} + 29q^{18} - 13q^{16} - 30q^{14} + 43q^{12} - q^{10} - 47q^8 + \\
&\quad + 40q^6 + 12q^4 - 46q^2 + 28 + 13q^{-2} - 31q^{-4} + 16q^{-6} + 7q^{-8} - 14q^{-10} + 6q^{-12} + 2q^{-14} - 3q^{-16} + q^{-18}) + \\
&\quad \left. + *S_{[2,1,1,1,1]}(-q^{24} + 3q^{20} - 5q^{16} + 6q^{12} - 7q^8 + 6q^4 - 5 + 3q^{-4} - q^{-8}) + *S_{[1,1,1,1,1,1]}q^{12} \right)
\end{aligned} \tag{259}$$

$$\begin{aligned}
&\quad + (-q^{24} + q^{22} + 4q^{20} - 5q^{18} - 4q^{16} + 13q^{14} - q^{12} - 14q^{10} + \\
&\quad + 10q^8 + 6q^6 - 12q^4 + 4q^2 + 4 - 6q^{-2} + q^{-4} + q^{-6} - q^{-8})A^{-2} + \\
&\quad + (q^{24} - 2q^{22} + 6q^{18} - 8q^{16} - 7q^{14} + 19q^{12} - 4q^{10} - 21q^8 + \\
&\quad + 18q^6 + 8q^4 - 18q^2 + 9 + 6q^{-2} - 8q^{-4} + 4q^{-6} + 2q^{-8} - 2q^{-10} + q^{-12}) + \\
&\quad + (-q^{20} + q^{18} + 3q^{16} - 6q^{14} - 2q^{12} + 14q^{10} - 6q^8 - 16q^6 + \\
&\quad + 16q^4 + 4q^2 - 16 + 6q^{-2} + 4q^{-4} - 6q^{-6} + q^{-8} + q^{-10} - q^{-12})A^2 + \\
&\quad + (q^{14} - 2q^{12} - q^{10} + 6q^8 - 3q^6 - 7q^4 + 9q^2 + 1 - 7q^{-2} + 4q^{-4} + q^{-6} - 2q^{-8} + q^{-10})A^4 \Big) = \\
&= \frac{A^4 q^{-8}}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(-q^{10} + 3q^8 - 5q^6 + 6q^4 - 7q^2 + 6 - 5q^{-2} + 3q^{-4} - q^{-6}) + *S_{[3,1,1,1]}q^4 + *S_{[2,2,2]}q^4 + \right. \\
&\quad + *S_{[2,2,1,1]}(q^{30} - 3q^{28} + 10q^{24} - 11q^{22} - 9q^{20} + 29q^{18} - 13q^{16} - 30q^{14} + 43q^{12} - q^{10} - 47q^8 + \\
&\quad + 40q^6 + 12q^4 - 46q^2 + 28 + 13q^{-2} - 31q^{-4} + 16q^{-6} + 7q^{-8} - 14q^{-10} + 6q^{-12} + 2q^{-14} - 3q^{-16} + q^{-18}) + \\
&\quad \left. + *S_{[2,1,1,1,1]}(-q^{24} + 3q^{20} - 5q^{16} + 6q^{12} - 7q^8 + 6q^4 - 5 + 3q^{-4} - q^{-8}) + *S_{[1,1,1,1,1,1]}q^{12} \right)
\end{aligned} \tag{260}$$

$$\begin{aligned}
\frac{H_{[2]}}{*S_{[2]}} &= A^4 q^8 \left((q^2 - 2 + q^{-2} + 3q^{-4} - 5q^{-6} + 2q^{-8} + 5q^{-10} - 6q^{-12} - q^{-14} + 5q^{-16} - 2q^{-18} - 2q^{-20} + q^{-22}) A^{-4} + \right. \\
&\quad + (-q^8 + q^6 + q^4 - 6q^2 + 4 + 4q^{-2} - 12q^{-4} + 6q^{-6} + 10q^{-8} - 14q^{-10} - \\
&\quad - q^{-12} + 13q^{-14} - 4q^{-16} - 5q^{-18} + 4q^{-20} + q^{-22} - q^{-24}) A^{-2} + \\
&\quad + (q^{12} - 2q^{10} + 2q^8 + 4q^6 - 8q^4 + 6q^2 + 9 - 18q^{-2} + 8q^{-4} + 18q^{-6} - \\
&\quad - 21q^{-8} - 4q^{-10} + 19q^{-12} - 7q^{-14} - 8q^{-16} + 6q^{-18} - 2q^{-22} + q^{-24}) + \\
&\quad + (-q^{12} + q^{10} + q^8 - 6q^6 + 4q^4 + 6q^2 - 16 + 4q^{-2} + 16q^{-4} - 16q^{-6} - \\
&\quad - 6q^{-8} + 14q^{-10} - 2q^{-12} - 6q^{-14} + 3q^{-16} + q^{-18} - q^{-20}) A^2 + \\
&\quad \left. + (q^{10} - 2q^8 + q^6 + 4q^4 - 7q^2 + 1 + 9q^{-2} - 7q^{-4} - 3q^{-6} + 6q^{-8} - q^{-10} - 2q^{-12} + q^{-14}) A^4 \right) = \\
&= \frac{A^4 q^8}{*S_{[2]}} \left(*S_{[6]} q^{-12} + *S_{[5,1]} (-q^8 + 3q^4 - 5 + 6q^{-4} - 7q^{-8} + 6q^{-12} - 5q^{-16} + 3q^{-20} - q^{-24}) + \right. \\
&\quad + *S_{[4,2]} (q^{18} - 3q^{16} + 2q^{14} + 6q^{12} - 14q^{10} + 7q^8 + 16q^6 - 31q^4 + 13q^2 + 28 - 46q^{-2} + 12q^{-4} + \\
&\quad + 40q^{-6} - 47q^{-8} - q^{-10} + 43q^{-12} - 30q^{-14} - 13q^{-16} + 29q^{-18} - 9q^{-20} - 11q^{-22} + 10q^{-24} - 3q^{-28} + q^{-30}) + \\
&\quad \left. + *S_{[4,1,1]} q^{-4} + *S_{[3,3]} q^{-4} + *S_{[3,2,1]} (-q^6 + 3q^4 - 5q^2 + 6 - 7q^{-2} + 6q^{-4} - 5q^{-6} + 3q^{-8} - q^{-10}) + *S_{[2,2,2]} \right)
\end{aligned} \tag{261}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = q^6 - 4q^4 + 8q^2 - 9 + 8q^{-2} - 4q^{-4} + q^{-6} \tag{262}$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = q^{12} - 4q^8 + 8q^4 - 9 + 8q^{-4} - 4q^{-8} + q^{-12} \tag{263}$$

Jones polynomials

$$J_{[1]} = -q^{12} + 3q^{10} - 5q^8 + 6q^6 - 6q^4 + 6q^2 - 4 + 3q^{-2} - q^{-4} \tag{264}$$

$$J_{[1,1]} = 1 \tag{265}$$

$$\begin{aligned}
J_{[2]} &= q^{34} - 3q^{32} + 2q^{30} + 6q^{28} - 15q^{26} + 7q^{24} + 19q^{22} - 32q^{20} + 8q^{18} + 32q^{16} - 41q^{14} + 4q^{12} + \\
&\quad + 38q^{10} - 37q^8 - 3q^6 + 35q^4 - 25q^2 - 8 + 24q^{-2} - 10q^{-4} - 8q^{-6} + 10q^{-8} - q^{-10} - 3q^{-12} + q^{-14}
\end{aligned} \tag{266}$$

Special polynomials

$$\mathfrak{H}_{[1]} = 2A^2 - A^4 \tag{267}$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 4A^4 - 4A^6 + A^8 \tag{268}$$

Ooguri-Vafa polynomials

$$\begin{aligned}
f_{[1,1]} &= \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2})}{\{q\}} \left((q^{13} - q^{11} - 3q^9 + q^7 + 3q^5 - 4q^3) A^2 + \right. \\
&\quad + (-q^9 - 2q^7 + 6q^5 - q^3 - q^1 - 2q^{-1} + 5q^{-3} - q^{-5} + q^{-9}) A^4 + \\
&\quad \left. + (-q^7 + q^5 + q^1 - 5q^{-1} + 2q^{-3} + 2q^{-5} - 3q^{-7} + q^{-11} - q^{-13}) A^6 \right)
\end{aligned} \tag{269}$$

$$\begin{aligned}
f_{[2]} &= \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2})}{\{q\}} \left((4q^1 - 3q^{-1} - q^{-3} + 3q^{-5} + q^{-7} - q^{-9}) A^2 + \right. \\
&\quad \left. + (-q^{13} + q^9 - 5q^7 + 2q^5 + q^3 + q^1 - 6q^{-1} + 2q^{-3} + q^{-5}) A^4 + (q^{17} - q^{15} + 3q^{11} - 2q^9 - 2q^7 + 5q^5 - q^3 - q^{-1} + q^{-3}) A^6 \right)
\end{aligned} \tag{270}$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = A^4 (3A^2 - 4 + 3A^{-2}) (A - A^{-1})^3 \tag{271}$$

Numbers $N_{R,n,k}$

$N_{[1]} :$

$k \backslash n =$	-1	1	3	5
0	0	-2	3	-1
1	2	-7	7	-2
2	1	-5	5	-1
3	0	-1	1	0

$N_{[1,1]} :$

$k \backslash n =$	-2	0	2	4	6	8	10
0	-3	-1	44	-106	109	-53	10
1	-4	-43	305	-685	721	-368	74
2	11	-145	779	-1824	2075	-1135	239
3	19	-178	1012	-2710	3408	-1958	407
4	8	-96	757	-2510	3493	-2052	400
5	1	-23	345	-1504	2294	-1347	234
6	0	-2	96	-579	957	-551	79
7	0	0	15	-137	243	-135	14
8	0	0	1	-18	34	-18	1
9	0	0	0	-1	2	-1	0

$N_{[2]} :$

$k \backslash n =$	-2	0	2	4	6	8	10
0	-6	15	7	-58	72	-37	7
1	-14	20	126	-395	447	-229	45
2	10	-87	441	-1013	1124	-598	123
3	43	-219	659	-1337	1530	-848	172
4	34	-189	499	-1013	1247	-710	132
5	10	-75	198	-456	624	-357	56
6	1	-14	39	-120	186	-104	12
7	0	-1	3	-17	30	-16	1
8	0	0	0	-1	2	-1	0

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$(2, -1|1, -1|1, -2)$

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= (q^4 - 2q^2 + 3 - 2q^{-2} + q^{-4})A^{-2} + \\
&+ (-q^6 + 2q^4 - 4q^2 + 5 - 4q^{-2} + 2q^{-4} - q^{-6}) + (q^4 - 2q^2 + 3 - 2q^{-2} + q^{-4})A^2 = \\
&= \frac{1}{*S_{[1]}} \left(*S_{[3]} + *S_{[2,1]}(q^8 - 3q^6 + 5q^4 - 7q^2 + 7 - 7q^{-2} + 5q^{-4} - 3q^{-6} + q^{-8}) + *S_{[1,1,1]} \right) \\
\frac{H_{[1,1]}}{*S_{[1,1]}} &= (q^{16} - 2q^{14} + 6q^{10} - 5q^8 - 5q^6 + 10q^4 - 2q^2 - 6 + 5q^{-2} - 2q^{-6} + q^{-8})A^{-4} + \\
&+ (-q^{18} + q^{16} + 2q^{14} - 7q^{12} + q^{10} + 13q^8 - 13q^6 - 10q^4 + 22q^2 - \\
&- 4 - 15q^{-2} + 11q^{-4} + 2q^{-6} - 6q^{-8} + 2q^{-10} + q^{-12} - q^{-14})A^{-2} + \\
&+ (q^{18} - 2q^{16} + q^{14} + 5q^{12} - 9q^{10} + q^8 + 18q^6 - 19q^4 - 10q^2 + 31 - \\
&- 10q^{-2} - 19q^{-4} + 18q^{-6} + q^{-8} - 9q^{-10} + 5q^{-12} + q^{-14} - 2q^{-16} + q^{-18}) + \\
&+ (-q^{14} + q^{12} + 2q^{10} - 6q^8 + 2q^6 + 11q^4 - 15q^2 - 4 + 22q^{-2} - \\
&- 10q^{-4} - 13q^{-6} + 13q^{-8} + q^{-10} - 7q^{-12} + 2q^{-14} + q^{-16} - q^{-18})A^2 + \\
&+ (q^8 - 2q^6 + 5q^2 - 6 - 2q^{-2} + 10q^{-4} - 5q^{-6} - 5q^{-8} + 6q^{-10} - 2q^{-14} + q^{-16})A^4 = \\
&= \frac{1}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(q^8 - 3q^6 + 5q^4 - 7q^2 + 7 - 7q^{-2} + 5q^{-4} - 3q^{-6} + q^{-8}) + *S_{[3,1,1,1]} + *S_{[2,2,2]} + \right. \\
&+ *S_{[2,2,1,1]}(q^{24} - 3q^{22} + q^{20} + 8q^{18} - 14q^{16} + 27q^{12} - 30q^{10} - 10q^8 + 53q^6 - 38q^4 - 28q^2 + \\
&+ 66 - 28q^{-2} - 38q^{-4} + 53q^{-6} - 10q^{-8} - 30q^{-10} + 27q^{-12} - 14q^{-16} + 8q^{-18} + q^{-20} - 3q^{-22} + q^{-24}) + \\
&+ *S_{[2,1,1,1,1]}(q^{16} - 3q^{12} + 5q^8 - 7q^4 + 7 - 7q^{-4} + 5q^{-8} - 3q^{-12} + q^{-16}) + *S_{[1,1,1,1,1,1]} \Big) \\
\frac{H_{[2]}}{*S_{[2]}} &= (q^8 - 2q^6 + 5q^2 - 6 - 2q^{-2} + 10q^{-4} - 5q^{-6} - 5q^{-8} + 6q^{-10} - 2q^{-14} + q^{-16})A^{-4} + \\
&+ (-q^{14} + q^{12} + 2q^{10} - 6q^8 + 2q^6 + 11q^4 - 15q^2 - 4 + 22q^{-2} - \\
&- 10q^{-4} - 13q^{-6} + 13q^{-8} + q^{-10} - 7q^{-12} + 2q^{-14} + q^{-16} - q^{-18})A^{-2} + \\
&+ (q^{18} - 2q^{16} + q^{14} + 5q^{12} - 9q^{10} + q^8 + 18q^6 - 19q^4 - 10q^2 + 31 - \\
&- 10q^{-2} - 19q^{-4} + 18q^{-6} + q^{-8} - 9q^{-10} + 5q^{-12} + q^{-14} - 2q^{-16} + q^{-18}) + \\
&+ (-q^{18} + q^{16} + 2q^{14} - 7q^{12} + q^{10} + 13q^8 - 13q^6 - 10q^4 + 22q^2 - \\
&- 4 - 15q^{-2} + 11q^{-4} + 2q^{-6} - 6q^{-8} + 2q^{-10} + q^{-12} - q^{-14})A^2 + \\
&+ (q^{16} - 2q^{14} + 6q^{10} - 5q^8 - 5q^6 + 10q^4 - 2q^2 - 6 + 5q^{-2} - 2q^{-6} + q^{-8})A^4 = \\
&= \frac{1}{*S_{[2]}} \left(*S_{[6]} + *S_{[5,1]}(q^{16} - 3q^{12} + 5q^8 - 7q^4 + 7 - 7q^{-4} + 5q^{-8} - 3q^{-12} + q^{-16}) + \right. \\
&+ *S_{[4,2]}(q^{24} - 3q^{22} + q^{20} + 8q^{18} - 14q^{16} + 27q^{12} - 30q^{10} - 10q^8 + 53q^6 - 38q^4 - 28q^2 + \\
&+ 66 - 28q^{-2} - 38q^{-4} + 53q^{-6} - 10q^{-8} - 30q^{-10} + 27q^{-12} - 14q^{-16} + 8q^{-18} + q^{-20} - 3q^{-22} + q^{-24}) + \\
&+ *S_{[4,1,1]} + *S_{[3,3]} + *S_{[3,2,1]}(q^8 - 3q^6 + 5q^4 - 7q^2 + 7 - 7q^{-2} + 5q^{-4} - 3q^{-6} + q^{-8}) + *S_{[2,2,2]} \Big)
\end{aligned}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = -q^6 + 4q^4 - 8q^2 + 11 - 8q^{-2} + 4q^{-4} - q^{-6} \quad (275)$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = -q^{12} + 4q^8 - 8q^4 + 11 - 8q^{-4} + 4q^{-8} - q^{-12} \quad (276)$$

Jones polynomials

$$J_{[1]} = q^8 - 3q^6 + 5q^4 - 6q^2 + 7 - 6q^{-2} + 5q^{-4} - 3q^{-6} + q^{-8} \quad (277)$$

$$J_{[1,1]} = 1 \quad (278)$$

$$\begin{aligned}
J_{[2]} &= q^{24} - 3q^{22} + q^{20} + 9q^{18} - 14q^{16} - 3q^{14} + 28q^{12} - 25q^{10} - 14q^8 + 47q^6 - 29q^4 - 25q^2 + \\
&+ 55 - 25q^{-2} - 29q^{-4} + 47q^{-6} - 14q^{-8} - 25q^{-10} + 28q^{-12} - 3q^{-14} - 14q^{-16} + 9q^{-18} + q^{-20} - 3q^{-22} + q^{-24}
\end{aligned} \quad (279)$$

Special polynomials

$$\mathfrak{H}_{[1]} = A^{-2} - 1 + A^2 \quad (280)$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = A^{-4} - 2A^{-2} + 3 - 2A^2 + A^4 \quad (281)$$

Ooguri-Vafa polynomials

$$f_{[1,1]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2})}{\{q\}} \left((q^{15} - q^{13} - q^{11} + 4q^9 - 3q^5 + 4q^1 - 2q^{-1}) A^{-2} + \right. \\ \left. + (-q^{11} + q^9 + 2q^7 - 8q^3 + 8q^1 - 2q^{-3} - q^{-5} + q^{-7}) + (2q^5 - 4q^3 + 3q^{-1} - 4q^{-5} + q^{-7} + q^{-9} - q^{-11}) A^2 \right) \quad (282)$$

$$f_{[2]} = \frac{\{A\}^2 \{A/q\} \{Aq\} (q^2 - 1 + q^{-2})}{\{q\}} \left((2q^5 - 4q^3 + 3q^{-1} - 4q^{-5} + q^{-7} + q^{-9} - q^{-11}) A^{-2} + \right. \\ \left. + (-q^{11} + q^9 + 2q^7 - 8q^3 + 8q^1 - 2q^{-3} - q^{-5} + q^{-7}) + (q^{15} - q^{13} - q^{11} + 4q^9 - 3q^5 + 4q^1 - 2q^{-1}) A^2 \right) \quad (283)$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = 2(A^2 - A^{-2})(A - A^{-1})^3 \quad (284)$$

Numbers $N_{R,n,k}$

$N_{[1]} :$

$k \setminus n =$	-3	-1	1	3
0	-1	2	-2	1
1	-2	7	-7	2
2	-1	5	-5	1
3	0	1	-1	0

$N_{[1,1]} :$

$k \setminus n =$	-6	-4	-2	0	2	4	6
0	6	-24	44	-56	54	-32	8
1	29	-128	256	-350	345	-198	46
2	56	-277	607	-899	916	-514	111
3	60	-322	753	-1223	1328	-749	153
4	36	-213	522	-959	1148	-658	124
5	10	-77	200	-444	601	-345	55
6	1	-14	39	-119	184	-103	12
7	0	-1	3	-17	30	-16	1
8	0	0	0	-1	2	-1	0

$N_{[2]} :$

$k \setminus n =$	-6	-4	-2	0	2	4	6
0	8	-32	54	-56	44	-24	6
1	46	-198	345	-350	256	-128	29
2	111	-514	916	-899	607	-277	56
3	153	-749	1328	-1223	753	-322	60
4	124	-658	1148	-959	522	-213	36
5	55	-345	601	-444	200	-77	10
6	12	-103	184	-119	39	-14	1
7	1	-16	30	-17	3	-1	0
8	0	-1	2	-1	0	0	0

Knot 8_{18}

$$(1, -1|1, -1|1, -1|1, -1)$$

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= (q^4 - 3q^2 + 3 - 3q^{-2} + q^{-4})A^{-2} + \\
&+ (-q^6 + 3q^4 - 4q^2 + 7 - 4q^{-2} + 3q^{-4} - q^{-6}) + (q^4 - 3q^2 + 3 - 3q^{-2} + q^{-4})A^2 = \\
&= \frac{1}{*S_{[1]}} \left(*S_{[3]} + *S_{[2,1]}(q^8 - 4q^6 + 6q^4 - 8q^2 + 9 - 8q^{-2} + 6q^{-4} - 4q^{-6} + q^{-8}) + *S_{[1,1,1]} \right) \\
\frac{H_{[1,1]}}{*S_{[1,1]}} &= (q^{16} - 3q^{14} + 9q^{10} - 8q^8 - 7q^6 + 15q^4 - 3q^2 - 9 + 8q^{-2} - 3q^{-6} + q^{-8})A^{-4} + \\
&+ (-q^{18} + 2q^{16} + 3q^{14} - 10q^{12} + 2q^{10} + 19q^8 - 20q^6 - 15q^4 + 31q^2 - \\
&- 8 - 23q^{-2} + 16q^{-4} + 3q^{-6} - 9q^{-8} + 3q^{-10} + 2q^{-12} - q^{-14})A^{-2} + \\
&+ (q^{18} - 3q^{16} + q^{14} + 7q^{12} - 14q^{10} + 2q^8 + 27q^6 - 26q^4 - 13q^2 + \\
&+ 47 - 13q^{-2} - 26q^{-4} + 27q^{-6} + 2q^{-8} - 14q^{-10} + 7q^{-12} + q^{-14} - 3q^{-16} + q^{-18}) + \\
&+ (-q^{14} + 2q^{12} + 3q^{10} - 9q^8 + 3q^6 + 16q^4 - 23q^2 - 8 + 31q^{-2} - 15q^{-4} - \\
&- 20q^{-6} + 19q^{-8} + 2q^{-10} - 10q^{-12} + 3q^{-14} + 2q^{-16} - q^{-18})A^2 + \\
&+ (q^8 - 3q^6 + 8q^2 - 9 - 3q^{-2} + 15q^{-4} - 7q^{-6} - 8q^{-8} + 9q^{-10} - 3q^{-14} + q^{-16})A^4 = \\
&= \frac{1}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(q^8 - 4q^6 + 6q^4 - 8q^2 + 9 - 8q^{-2} + 6q^{-4} - 4q^{-6} + q^{-8}) + *S_{[3,1,1,1]} + \right. \\
&+ *S_{[2,2,2]} + *S_{[2,2,1,1]}(q^{24} - 4q^{22} + 2q^{20} + 12q^{18} - 21q^{16} + 40q^{12} - 44q^{10} - 15q^8 + 76q^6 - 54q^4 - 40q^2 + \\
&+ 94 - 40q^{-2} - 54q^{-4} + 76q^{-6} - 15q^{-8} - 44q^{-10} + 40q^{-12} - 21q^{-16} + 12q^{-18} + 2q^{-20} - 4q^{-22} + q^{-24}) + \\
&\left. + *S_{[2,1,1,1,1]}(q^{16} - 4q^{12} + 6q^8 - 8q^4 + 9 - 8q^{-4} + 6q^{-8} - 4q^{-12} + q^{-16}) + *S_{[1,1,1,1,1,1]} \right) \\
\frac{H_{[2]}}{*S_{[2]}} &= (q^8 - 3q^6 + 8q^2 - 9 - 3q^{-2} + 15q^{-4} - 7q^{-6} - 8q^{-8} + 9q^{-10} - 3q^{-14} + q^{-16})A^{-4} + \\
&+ (-q^{14} + 2q^{12} + 3q^{10} - 9q^8 + 3q^6 + 16q^4 - 23q^2 - 8 + 31q^{-2} - 15q^{-4} - \\
&- 20q^{-6} + 19q^{-8} + 2q^{-10} - 10q^{-12} + 3q^{-14} + 2q^{-16} - q^{-18})A^{-2} + \\
&+ (q^{18} - 3q^{16} + q^{14} + 7q^{12} - 14q^{10} + 2q^8 + 27q^6 - 26q^4 - 13q^2 + \\
&+ 47 - 13q^{-2} - 26q^{-4} + 27q^{-6} + 2q^{-8} - 14q^{-10} + 7q^{-12} + q^{-14} - 3q^{-16} + q^{-18}) + \\
&+ (-q^{18} + 2q^{16} + 3q^{14} - 10q^{12} + 2q^{10} + 19q^8 - 20q^6 - 15q^4 + 31q^2 - \\
&- 8 - 23q^{-2} + 16q^{-4} + 3q^{-6} - 9q^{-8} + 3q^{-10} + 2q^{-12} - q^{-14})A^2 + \\
&+ (q^{16} - 3q^{14} + 9q^{10} - 8q^8 - 7q^6 + 15q^4 - 3q^2 - 9 + 8q^{-2} - 3q^{-6} + q^{-8})A^4 = \\
&= \frac{1}{*S_{[2]}} \left(*S_{[6]} + *S_{[5,1]}(q^{16} - 4q^{12} + 6q^8 - 8q^4 + 9 - 8q^{-4} + 6q^{-8} - 4q^{-12} + q^{-16}) + \right. \\
&+ *S_{[4,2]}(q^{24} - 4q^{22} + 2q^{20} + 12q^{18} - 21q^{16} + 40q^{12} - 44q^{10} - 15q^8 + 76q^6 - 54q^4 - 40q^2 + \\
&+ 94 - 40q^{-2} - 54q^{-4} + 76q^{-6} - 15q^{-8} - 44q^{-10} + 40q^{-12} - 21q^{-16} + 12q^{-18} + 2q^{-20} - 4q^{-22} + q^{-24}) + \\
&\left. + *S_{[4,1,1]} + *S_{[3,3]} + *S_{[3,2,1]}(q^8 - 4q^6 + 6q^4 - 8q^2 + 9 - 8q^{-2} + 6q^{-4} - 4q^{-6} + q^{-8}) + *S_{[2,2,2]} \right)
\end{aligned}$$

$$\begin{aligned}
&+ (-q^{14} + 2q^{12} + 3q^{10} - 9q^8 + 3q^6 + 16q^4 - 23q^2 - 8 + 31q^{-2} - 15q^{-4} - \\
&- 20q^{-6} + 19q^{-8} + 2q^{-10} - 10q^{-12} + 3q^{-14} + 2q^{-16} - q^{-18})A^{-2} + \\
&+ (q^{18} - 3q^{16} + q^{14} + 7q^{12} - 14q^{10} + 2q^8 + 27q^6 - 26q^4 - 13q^2 + \\
&+ 47 - 13q^{-2} - 26q^{-4} + 27q^{-6} + 2q^{-8} - 14q^{-10} + 7q^{-12} + q^{-14} - 3q^{-16} + q^{-18}) + \\
&+ (-q^{18} + 2q^{16} + 3q^{14} - 10q^{12} + 2q^{10} + 19q^8 - 20q^6 - 15q^4 + 31q^2 - \\
&- 8 - 23q^{-2} + 16q^{-4} + 3q^{-6} - 9q^{-8} + 3q^{-10} + 2q^{-12} - q^{-14})A^2 + \\
&+ (q^{16} - 3q^{14} + 9q^{10} - 8q^8 - 7q^6 + 15q^4 - 3q^2 - 9 + 8q^{-2} - 3q^{-6} + q^{-8})A^4 = \\
&= \frac{1}{*S_{[2]}} \left(*S_{[6]} + *S_{[5,1]}(q^{16} - 4q^{12} + 6q^8 - 8q^4 + 9 - 8q^{-4} + 6q^{-8} - 4q^{-12} + q^{-16}) + \right. \\
&+ *S_{[4,2]}(q^{24} - 4q^{22} + 2q^{20} + 12q^{18} - 21q^{16} + 40q^{12} - 44q^{10} - 15q^8 + 76q^6 - 54q^4 - 40q^2 + \\
&+ 94 - 40q^{-2} - 54q^{-4} + 76q^{-6} - 15q^{-8} - 44q^{-10} + 40q^{-12} - 21q^{-16} + 12q^{-18} + 2q^{-20} - 4q^{-22} + q^{-24}) + \\
&\left. + *S_{[4,1,1]} + *S_{[3,3]} + *S_{[3,2,1]}(q^8 - 4q^6 + 6q^4 - 8q^2 + 9 - 8q^{-2} + 6q^{-4} - 4q^{-6} + q^{-8}) + *S_{[2,2,2]} \right)
\end{aligned}$$

Alexander polynomials

$$\begin{aligned}
\mathfrak{A}_{[1]} &= -q^6 + 5q^4 - 10q^2 + 13 - 10q^{-2} + 5q^{-4} - q^{-6} \\
\mathfrak{A}_{[1,1]} &= \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = -q^{12} + 5q^8 - 10q^4 + 13 - 10q^{-4} + 5q^{-8} - q^{-12}
\end{aligned}$$

Jones polynomials

$$J_{[1]} = q^8 - 4q^6 + 6q^4 - 7q^2 + 9 - 7q^{-2} + 6q^{-4} - 4q^{-6} + q^{-8} \quad (289)$$

$$J_{[1,1]} = 1 \quad (290)$$

$$\begin{aligned}
J_{[2]} &= q^{24} - 4q^{22} + 2q^{20} + 13q^{18} - 21q^{16} - 4q^{14} + 41q^{12} - 38q^{10} - 20q^8 + 69q^6 - 43q^4 - 36q^2 + \\
&+ 81 - 36q^{-2} - 43q^{-4} + 69q^{-6} - 20q^{-8} - 38q^{-10} + 41q^{-12} - 4q^{-14} - 21q^{-16} + 13q^{-18} + 2q^{-20} - 4q^{-22} + q^{-24}
\end{aligned}$$

Special polynomials

$$\mathfrak{H}_{[1]} = -A^{-2} + 3 - A^2 \quad (292)$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = A^{-4} - 6A^{-2} + 11 - 6A^2 + A^4 \quad (293)$$

Ooguri-Vafa polynomials

$$f_{[1,1]} = \frac{\{A\}^2\{A/q\}\{Aq\}}{\{q\}} \left((q^{15} - 3q^{13} + q^{11} + 6q^9 - 8q^7 + 2q^5 + 4q^3 + 2q^1 - 9q^{-1} + 9q^{-3} - 3q^{-5})A^{-2} + \right. \\ \left. + (-q^{11} + 3q^9 - 2q^5 - 9q^3 + 23q^1 - 23q^{-1} + 9q^{-3} + 2q^{-5} - 3q^{-9} + q^{-11}) + \right. \\ \left. + (3q^5 - 9q^3 + 9q^1 - 2q^{-1} - 4q^{-3} - 2q^{-5} + 8q^{-7} - 6q^{-9} - q^{-11} + 3q^{-13} - q^{-15})A^2 \right) \quad (294)$$

$$f_{[2]} = \frac{\{A\}^2\{A/q\}\{Aq\}}{\{q\}} \left((3q^5 - 9q^3 + 9q^1 - 2q^{-1} - 4q^{-3} - 2q^{-5} + 8q^{-7} - 6q^{-9} - q^{-11} + 3q^{-13} - q^{-15})A^{-2} + \right. \\ \left. + (-q^{11} + 3q^9 - 2q^5 - 9q^3 + 23q^1 - 23q^{-1} + 9q^{-3} + 2q^{-5} - 3q^{-9} + q^{-11}) + \right. \\ \left. + (q^{15} - 3q^{13} + q^{11} + 6q^9 - 8q^7 + 2q^5 + 4q^3 + 2q^1 - 9q^{-1} + 9q^{-3} - 3q^{-5})A^2 \right) \quad (295)$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = 2(A^2 - A^{-2})(A - A^{-1})^3 \quad (296)$$

Numbers $N_{R,n,k}$

$N_{[1]} :$

$k \setminus n =$	-3	-1	1	3
0	1	-4	4	-1
1	-1	2	-2	1
2	-1	4	-4	1
3	0	1	-1	0

$N_{[1,1]} :$

$k \setminus n =$	-6	-4	-2	0	2	4	6
0	2	-6	8	-12	18	-14	4
1	4	-2	-32	54	-18	-12	6
2	10	-42	54	-46	58	-44	10
3	26	-131	288	-406	388	-211	46
4	26	-138	328	-542	593	-338	71
5	9	-63	161	-327	421	-244	43
6	1	-13	36	-102	154	-87	11
7	0	-1	3	-16	28	-15	1
8	0	0	0	-1	2	-1	0

$N_{[2]} :$

$k \setminus n =$	-6	-4	-2	0	2	4	6
0	4	-14	18	-12	8	-6	2
1	6	-12	-18	54	-32	-2	4
2	10	-44	58	-46	54	-42	10
3	46	-211	388	-406	288	-131	26
4	71	-338	593	-542	328	-138	26
5	43	-244	421	-327	161	-63	9
6	11	-87	154	-102	36	-13	1
7	1	-15	28	-16	3	-1	0
8	0	-1	2	-1	0	0	0

Knot 8₁₉

$$(1, 3|1, 3) = (1, 1|1, 1|1, 1|1, 1)$$

HOMFLY polynomials

$$\frac{H_{[1]}}{*S_{[1]}} = A^{-8} \left(A^{-2} + (-q^4 - q^2 - 1 - q^{-2} - q^{-4}) + (q^6 + q^2 + 1 + q^{-2} + q^{-6})A^2 \right) =$$

$$= \frac{A^{-8}}{*S_{[1]}} \left(*S_{[3]}q^8 - *S_{[2,1]} + *S_{[1,1,1]}q^{-8} \right)$$

(297)

$$\begin{aligned} \frac{H_{[1,1]}}{*S_{[1,1]}} &= A^{-16}q^{32} \left(q^{-16}A^{-4} + (-q^{-12} - 2q^{-14} - q^{-16} - q^{-18} - 2q^{-20} - q^{-22} - q^{-24} - q^{-26})A^{-2} + \right. \\ &+ (2q^{-10} + 2q^{-12} + 3q^{-14} + 3q^{-16} + 5q^{-18} + 4q^{-20} + 4q^{-22} + 3q^{-24} + 3q^{-26} + 2q^{-28} + 2q^{-30} + q^{-32} + q^{-34}) + \\ &+ (-q^{-8} - 2q^{-10} - 2q^{-12} - 3q^{-14} - 5q^{-16} - 5q^{-18} - 4q^{-20} - 5q^{-22} - 5q^{-24} - \\ &- 4q^{-26} - 4q^{-28} - 3q^{-30} - 2q^{-32} - 2q^{-34} - q^{-36} - q^{-38} - q^{-40})A^2 + \\ &+ (q^{-8} + q^{-12} + 2q^{-14} + 2q^{-16} + q^{-18} + 3q^{-20} + 2q^{-22} + 2q^{-24} + 2q^{-26} + \\ &+ 2q^{-28} + q^{-30} + 2q^{-32} + q^{-34} + q^{-36} + q^{-38} + q^{-44})A^4 \Big) = \end{aligned}$$

(298)

$$= \frac{A^{-16}q^{32}}{*S_{[1,1]}} \left(*S_{[3,3]} - *S_{[3,2,1]}q^{-8} + *S_{[3,1,1,1]}q^{-16} + *S_{[2,2,2]}q^{-16} - *S_{[2,1,1,1,1]}q^{-32} + *S_{[1,1,1,1,1,1]}q^{-48} \right)$$

$$\begin{aligned} \frac{H_{[2]}}{*S_{[2]}} &= A^{-16}q^{-32} \left(q^{16}A^{-4} + (-q^{26} - q^{24} - q^{22} - 2q^{20} - q^{18} - q^{16} - 2q^{14} - q^{12})A^{-2} + \right. \\ &+ (q^{34} + q^{32} + 2q^{30} + 2q^{28} + 3q^{26} + 3q^{24} + 4q^{22} + 4q^{20} + 5q^{18} + 3q^{16} + 3q^{14} + 2q^{12} + 2q^{10}) + \\ &+ (-q^{40} - q^{38} - q^{36} - 2q^{34} - 2q^{32} - 3q^{30} - 4q^{28} - 4q^{26} - 5q^{24} - 5q^{22} - 4q^{20} - 5q^{18} - 5q^{16} - 3q^{14} - 2q^{12} - 2q^{10} - q^8)A^2 + \\ &+ (q^{44} + q^{38} + q^{36} + q^{34} + 2q^{32} + q^{30} + 2q^{28} + 2q^{26} + 2q^{24} + 2q^{22} + 3q^{20} + q^{18} + 2q^{16} + 2q^{14} + q^{12} + q^8)A^4 \Big) = \\ &= \frac{A^{-16}q^{-32}}{*S_{[2]}} \left(*S_{[6]}q^{48} - *S_{[5,1]}q^{32} + *S_{[4,1,1]}q^{16} + *S_{[3,3]}q^{16} - *S_{[3,2,1]}q^8 + *S_{[2,2,2]} \right) \end{aligned}$$

(299)

Alexander polynomials

$$\mathfrak{A}_{[1]} = q^6 - q^4 + 1 - q^{-4} + q^{-6}$$

(300)

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = q^{12} - q^8 + 1 - q^{-8} + q^{-12}$$

(301)

Jones polynomials

$$J_{[1]} = q^{-6} + q^{-10} - q^{-16}$$

(302)

$$J_{[1,1]} = 1$$

(303)

$$J_{[2]} = q^{-12} + q^{-18} + q^{-24} - q^{-26} - q^{-32} - q^{-38} + q^{-40} - q^{-44} + q^{-46}$$

(304)

Special polynomials

$$\mathfrak{H}_{[1]} = A^{-10} - 5A^{-8} + 5A^{-6}$$

(305)

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = A^{-20} - 10A^{-18} + 35A^{-16} - 50A^{-14} + 25A^{-12}$$

(306)

Ooguri-Vafa polynomials

$$\begin{aligned} f_{[1,1]} &= \frac{\{A\}^2\{A/q\}\{Aq\}(q^2 + q^{-2})}{\{q\}} \left((q^{13} + q^5)A^{-18} + (-q^{17} - q^{15} - q^{11} - 3q^9 - 2q^7 - q^5 - 2q^3 - 2q^1 - q^{-1})A^{-16} + \right. \\ &+ (q^{19} + q^{15} + 2q^{13} + 3q^{11} + q^9 + 3q^7 + 3q^5 + 3q^3 + q^1 + 2q^{-1} + q^{-3} + q^{-5})A^{-14} \Big) \end{aligned}$$

(307)

$$\begin{aligned} f_{[2]} &= \frac{\{A\}^2\{A/q\}\{Aq\}(q^2 + q^{-2})}{\{q\}} \left((-q^{-5} - q^{-13})A^{-18} + \right. \\ &+ (q^1 + 2q^{-1} + 2q^{-3} + q^{-5} + 2q^{-7} + 3q^{-9} + q^{-11} + q^{-15} + q^{-17})A^{-16} + \\ &+ (-q^5 - q^3 - 2q^1 - q^{-1} - 3q^{-3} - 3q^{-5} - 3q^{-7} - q^{-9} - 3q^{-11} - 2q^{-13} - q^{-15} - q^{-19})A^{-14} \Big) \end{aligned}$$

(308)

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = -\frac{4(11A^4 - 7A^2 + 1)(A - A^{-1})^3}{A^{18}}$$

(309)

Numbers $N_{R,n,k}$

$N_{[1]} :$

$k \backslash n =$	-11	-9	-7	-5
0	-1	6	-10	5
1	0	5	-15	10
2	0	1	-7	6
3	0	0	-1	1

$N_{[1,1]} :$

$k \backslash n =$	-22	-20	-18	-16	-14	-12	-10
0	16	-146	540	-1020	1040	-546	116
1	80	-775	3060	-6090	6410	-3395	710
2	148	-1709	7503	-15954	17363	-9219	1868
3	128	-2001	10254	-23804	26999	-14365	2789
4	56	-1365	8580	-22281	26597	-14182	2595
5	12	-560	4570	-13657	17333	-9247	1549
6	1	-136	1556	-5557	7575	-4029	590
7	0	-18	328	-1485	2196	-1159	138
8	0	-1	39	-250	405	-211	18
9	0	0	2	-24	43	-22	1
10	0	0	0	-1	2	-1	0

$k \setminus n =$	-22	-20	-18	-16	-14	-12	-10
0	20	-190	720	-1380	1420	-750	160
1	130	-1245	4890	-9710	10220	-5425	1140
2	314	-3403	14459	-30206	32619	-17325	3542
3	367	-4996	24019	-53906	60220	-32020	6316
4	230	-4367	24779	-61035	71182	-37920	7131
5	79	-2380	16653	-46053	56575	-30177	5303
6	14	-816	7422	-23680	30910	-16472	2622
7	1	-171	2177	-8320	11638	-6175	850
8	0	-20	404	-1963	2967	-1561	173
9	0	-1	43	-297	489	-254	20
10	0	0	2	-26	47	-24	1
11	0	0	0	-1	2	-1	0

$N_{[2]} :$

Knot 8_{20}

$(-1, -3 | -1, 3)$

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= A^2 \left((-q^2 + 1 - q^{-2})A^{-2} + (q^4 + 2 + q^{-4}) + (-q^2 - q^{-2})A^2 \right) = \\
&= \frac{A^2}{*S_{[1]}} \left(*S_{[3]}q^{-2} + *S_{[2,1]}(-q^6 + q^4 - q^2 + 1 - q^{-2} + q^{-4} - q^{-6}) + *S_{[1,1,1]}q^2 \right) \\
\frac{H_{[1,1]}}{*S_{[1,1]}} &= A^4 q^{-8} \left((-q^{12} + q^{10} + q^8 - q^6 + q^2)A^{-4} + \right. \\
&+ (-q^{12} - q^{10} + q^8 - q^6 - 3q^4 - q^2 - q^{-2} - q^{-4})A^{-2} + (q^{16} + q^{12} + q^{10} + 2q^8 + 3q^6 + 3q^4 + q^2 + 3 + 3q^{-2} + q^{-4} + q^{-8}) + \\
&+ (-q^{12} - q^{10} - q^8 - 2q^6 - 2q^4 - 2q^2 - 3 - 2q^{-2} - q^{-6} - q^{-8})A^2 + (q^6 + q^2 + 1 + q^{-6})A^4 \Big) = \\
&= \frac{A^4 q^{-8}}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(-q^8 + q^6 - q^4 + q^2 - 1 + q^{-2} - q^{-4}) + *S_{[3,1,1,1]}q^4 + *S_{[2,2,2]}q^4 + \right. \\
&+ *S_{[2,2,1,1]}(q^{22} - q^{20} + q^{16} - q^{14} - q^2 + 1 + q^{-2} - 2q^{-4} + 2q^{-8} - q^{-10} - q^{-12} + q^{-14}) + \\
&\left. + *S_{[2,1,1,1,1]}(-q^{20} + q^{16} - q^{12} + q^8 - q^4 + 1 - q^{-4}) + *S_{[1,1,1,1,1,1]}q^{12} \right)
\end{aligned} \tag{310}$$

(311)

$$\begin{aligned}
\frac{H_{[2]}}{*S_{[2]}} &= A^4 q^8 \left((q^{-2} - q^{-6} + q^{-8} + q^{-10} - q^{-12}) A^{-4} + (-q^4 - q^2 - q^{-2} - 3q^{-4} - q^{-6} + q^{-8} - q^{-10} - q^{-12}) A^{-2} + \right. \\
&\quad \left. + (q^8 + q^4 + 3q^2 + 3 + q^{-2} + 3q^{-4} + 3q^{-6} + 2q^{-8} + q^{-10} + q^{-12} + q^{-16}) + \right. \\
&\quad \left. + (-q^8 - q^6 - 2q^2 - 3 - 2q^{-2} - 2q^{-4} - 2q^{-6} - q^{-8} - q^{-10} - q^{-12}) A^2 + (q^6 + 1 + q^{-2} + q^{-6}) A^4 \right) = \\
&= \frac{A^4 q^8}{*S_{[2]}} \left(*S_{[6]} q^{-12} + *S_{[5,1]} (-q^4 + 1 - q^{-4} + q^{-8} - q^{-12} + q^{-16} - q^{-20}) + \right. \\
&\quad \left. + *S_{[4,2]} (q^{14} - q^{12} - q^{10} + 2q^8 - 2q^4 + q^2 + 1 - q^{-2} - q^{-14} + q^{-16} - q^{-20} + q^{-22}) + *S_{[4,1,1]} q^{-4} + *S_{[3,3]} q^{-4} + \right. \\
&\quad \left. + *S_{[3,2,1]} (-q^4 + q^2 - 1 + q^{-2} - q^{-4} + q^{-6} - q^{-8}) + *S_{[2,2,2]} \right)
\end{aligned} \tag{312}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = q^8 - 2q^4 + 3 - 2q^{-4} + q^{-8} \tag{313}$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = q^8 - 2q^4 + 3 - 2q^{-4} + q^{-8} \tag{314}$$

Jones polynomials

$$J_{[1]} = -q^{10} + q^8 - q^6 + 2q^4 - q^2 + 2 - q^{-2} \tag{315}$$

$$J_{[1,1]} = 1 \tag{316}$$

$$J_{[2]} = q^{30} - q^{28} - q^{26} + 2q^{24} - q^{22} - 2q^{20} + 2q^{18} - 2q^{14} + 2q^{12} + q^{10} - 2q^8 + q^6 + 2q^4 - 2q^2 + 1 + q^{-2} - q^{-4} \tag{317}$$

Special polynomials

$$\mathfrak{H}_{[1]} = -1 + 4A^2 - 2A^4 \tag{318}$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 1 - 8A^2 + 20A^4 - 16A^6 + 4A^8 \tag{319}$$

Ooguri-Vafa polynomials

$$\begin{aligned}
f_{[1,1]} &= \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left((-q^3 + q^1 - q^{-1}) A^2 + \right. \\
&\quad \left. + (q^1 + q^{-1} + q^{-3} + q^{-5} + q^{-7} + q^{-9}) A^4 + (-q^3 - 2q^{-1} - q^{-3} - 2q^{-5} - q^{-7} - q^{-9} - q^{-13}) A^6 \right)
\end{aligned} \tag{320}$$

$$\begin{aligned}
f_{[2]} &= \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left((q^1 - q^{-1} + q^{-3}) A^2 + \right. \\
&\quad \left. + (-q^9 - q^7 - q^5 - q^3 - q^1 - q^{-1}) A^4 + (q^{13} + q^9 + q^7 + 2q^5 + q^3 + 2q^1 + q^{-3}) A^6 \right)
\end{aligned} \tag{321}$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = A^4 (3A^2 - A^{-2})^2 (A - A^{-1})^3 \tag{322}$$

Numbers $N_{R,n,k}$

$N_{[1]} :$	$k \setminus n =$	-1	1	3	5
	0	1	-5	6	-2
	1	1	-5	5	-1
	2	0	-1	1	0

$k \setminus n =$	-2	0	2	4	6	8	10
0	0	-15	85	-190	210	-115	25
1	0	-35	250	-645	775	-440	95
2	0	-28	302	-953	1267	-743	155
3	0	-9	193	-781	1148	-680	129
4	0	-1	69	-377	607	-354	56
5	0	0	13	-106	185	-104	12
6	0	0	1	-16	30	-16	1
7	0	0	0	-1	2	-1	0

$N_{[1,1]} :$

$k \setminus n =$	-2	0	2	4	6	8	10
0	0	-5	46	-114	131	-73	16
1	0	-10	111	-319	400	-231	50
2	0	-6	104	-373	521	-309	63
3	0	-1	48	-231	359	-212	37
4	0	0	11	-79	135	-77	10
5	0	0	1	-14	26	-14	1
6	0	0	0	-1	2	-1	0

$N_{[2]} :$

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$(-2, 2 | -1, -3)$

HOMFLY polynomials

$$\frac{H_{[1]}}{*S_{[1]}} = A^4 \left((2q^2 - 1 + 2q^{-2})A^{-2} + (-q^4 + q^2 - 3 + q^{-2} - q^{-4}) + (q^2 - 1 + q^{-2})A^2 \right) = \quad (323)$$

$$= \frac{A^4}{*S_{[1]}} \left(*S_{[3]}q^{-4} + *S_{[2,1]}(q^6 - 2q^4 + 2q^2 - 3 + 2q^{-2} - 2q^{-4} + q^{-6}) + *S_{[1,1,1]}q^4 \right)$$

$$\begin{aligned} & \frac{H_{[1,1]}}{*S_{[1,1]}} = A^8 q^{-16} \left((q^{22} + 3q^{20} - 2q^{18} + 5q^{14} - q^{10} + 3q^8)A^{-4} + \right. \\ & \quad + (-q^{24} - 2q^{22} + q^{20} - 8q^{16} - 2q^{14} + 4q^{12} - 6q^{10} - 4q^8 + 2q^6 - q^4 - q^2)A^{-2} + \\ & \quad + (q^{22} + 3q^{18} + 3q^{16} - 3q^{14} + 3q^{12} + 8q^{10} - 3q^8 - q^6 + 4q^4 - 1 + q^{-2}) + \\ & \quad \left. + (-q^{18} - 4q^{12} + 3q^8 - 3q^6 - 2q^4 + 2q^2 - q^{-2})A^2 + (q^{12} - q^{10} + 2q^6 - q^4 - q^2 + 1)A^4 \right) = \quad (324) \end{aligned}$$

$$\begin{aligned} & = \frac{A^8 q^{-16}}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(q^{10} - 2q^8 + 2q^6 - 3q^4 + 2q^2 - 2 + q^{-2}) + *S_{[3,1,1,1]}q^8 + *S_{[2,2,2]}q^8 + \right. \\ & \quad + *S_{[2,2,1,1]}(q^{28} - 2q^{26} + 3q^{22} - 5q^{20} + 2q^{18} + 5q^{16} - 8q^{14} + 3q^{12} + \\ & \quad + 7q^{10} - 9q^8 + 8q^4 - 5q^2 - 3 + 5q^{-2} - q^{-4} - 2q^{-6} + q^{-8}) + \\ & \quad \left. + *S_{[2,1,1,1,1]}(q^{28} - 2q^{24} + 2q^{20} - 3q^{16} + 2q^{12} - 2q^8 + q^4) + *S_{[1,1,1,1,1,1]}q^{24} \right) \end{aligned}$$

$$\begin{aligned} & \frac{H_{[2]}}{*S_{[2]}} = A^8 q^{16} \left((3q^{-8} - q^{-10} + 5q^{-14} - 2q^{-18} + 3q^{-20} + q^{-22})A^{-4} + \right. \\ & \quad + (-q^{-2} - q^{-4} + 2q^{-6} - 4q^{-8} - 6q^{-10} + 4q^{-12} - 2q^{-14} - 8q^{-16} + q^{-20} - 2q^{-22} - q^{-24})A^{-2} + \\ & \quad + (q^2 - 1 + 4q^{-4} - q^{-6} - 3q^{-8} + 8q^{-10} + 3q^{-12} - 3q^{-14} + 3q^{-16} + 3q^{-18} + q^{-22}) + \\ & \quad + (-q^2 + 2q^{-2} - 2q^{-4} - 3q^{-6} + 3q^{-8} - 4q^{-12} - q^{-18})A^2 + \\ & \quad \left. + (1 - q^{-2} - q^{-4} + 2q^{-6} - q^{-10} + q^{-12})A^4 \right) = \quad (325) \end{aligned}$$

$$\begin{aligned}
&= \frac{A^8 q^{16}}{*S_{[2]}} \left(*S_{[6]} q^{-24} + *S_{[5,1]} (q^{-4} - 2q^{-8} + 2q^{-12} - 3q^{-16} + 2q^{-20} - 2q^{-24} + q^{-28}) + \right. \\
&\quad + *S_{[4,2]} (q^8 - 2q^6 - q^4 + 5q^2 - 3 - 5q^{-2} + 8q^{-4} - 9q^{-8} + 7q^{-10} + \\
&\quad + 3q^{-12} - 8q^{-14} + 5q^{-16} + 2q^{-18} - 5q^{-20} + 3q^{-22} - 2q^{-26} + q^{-28}) + \\
&\quad \left. + *S_{[4,1,1]} q^{-8} + *S_{[3,3]} q^{-8} + *S_{[3,2,1]} (q^2 - 2 + 2q^{-2} - 3q^{-4} + 2q^{-6} - 2q^{-8} + q^{-10}) + *S_{[2,2,2]} \right)
\end{aligned}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = -q^4 + 4q^2 - 5 + 4q^{-2} - q^{-4} \quad (326)$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = -q^8 + 4q^4 - 5 + 4q^{-4} - q^{-8} \quad (327)$$

Jones polynomials

$$J_{[1]} = q^{14} - 2q^{12} + 2q^{10} - 3q^8 + 3q^6 - 2q^4 + 2q^2 \quad (328)$$

$$J_{[1,1]} = 1 \quad (329)$$

$$\begin{aligned}
J_{[2]} &= q^{40} - 2q^{38} - q^{36} + 5q^{34} - 3q^{32} - 4q^{30} + 8q^{28} - 2q^{26} - 8q^{24} + \\
&\quad + 10q^{22} - q^{20} - 10q^{18} + 10q^{16} - 8q^{12} + 6q^{10} + q^8 - 4q^6 + 2q^4 + q^2
\end{aligned} \quad (330)$$

Special polynomials

$$\mathfrak{H}_{[1]} = 3A^2 - 3A^4 + A^6 \quad (331)$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = 9A^4 - 18A^6 + 15A^8 - 6A^{10} + A^{12} \quad (332)$$

Ooguri-Vafa polynomials

$$\begin{aligned}
f_{[1,1]} &= \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left((q^5 - 2q^{-1} + 2q^{-3} - 3q^{-5}) A^6 + \right. \\
&\quad \left. + (q^5 + 2q^1 - 2q^{-1} + 3q^{-3} + q^{-5} - q^{-7} + q^{-9} + q^{-11}) A^8 + (-q^{-1} - q^{-9} + q^{-13} - q^{-15}) A^{10} \right)
\end{aligned} \quad (333)$$

$$\begin{aligned}
f_{[2]} &= \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left((3q^5 - 2q^3 + 2q^1 - q^{-5}) A^6 + \right. \\
&\quad \left. + (-q^{11} - q^9 + q^7 - q^5 - 3q^3 + 2q^1 - 2q^{-1} - q^{-5}) A^8 + (q^{15} - q^{13} + q^9 + q^1) A^{10} \right)
\end{aligned} \quad (334)$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = 2A^8(A^2 - 3 + A^{-2})(A - A^{-1})^3 \quad (335)$$

Numbers $N_{R,n,k}$

$k \backslash n =$	1	3	5	7
0	-3	6	-4	1
1	-2	5	-4	1
2	0	1	-1	0

$N_{[1]} :$

$k \setminus n =$	2	4	6	8	10	12	14
0	9	-48	109	-136	99	-40	7
1	11	-104	348	-582	529	-250	48
2	3	-95	526	-1193	1320	-708	147
3	0	-46	468	-1424	1846	-1062	218
4	0	-11	253	-1034	1519	-893	166
5	0	-1	81	-458	743	-431	66
6	0	0	14	-120	211	-118	13
7	0	0	1	-17	32	-17	1
8	0	0	0	-1	2	-1	0

$N_{[1,1]} :$

$k \setminus n =$	2	4	6	8	10	12	14
0	7	-34	71	-84	61	-26	5
1	7	-58	185	-314	301	-152	31
2	1	-36	219	-542	642	-360	76
3	0	-10	148	-521	723	-425	85
4	0	-1	58	-289	453	-266	45
5	0	0	12	-91	157	-89	11
6	0	0	1	-15	28	-15	1
7	0	0	0	-1	2	-1	0

$N_{[2]} :$

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(2, 3|1, 4)

HOMFLY polynomials

$$\begin{aligned}
\frac{H_{[1]}}{*S_{[1]}} &= A^{-10} \left((q^2 - 1 + q^{-2})A^{-2} + (-q^6 - q^4 - 2 - q^{-4} - q^{-6}) + (q^8 + q^4 + q^2 + q^{-2} + q^{-4} + q^{-8})A^2 \right) = \\
&= \frac{A^{-10}}{*S_{[1]}} \left(*S_{[3]}q^{10} + *S_{[2,1]}(-q^4 + q^2 - 1 + q^{-2} - q^{-4}) + *S_{[1,1,1]}q^{-10} \right) \tag{336} \\
\frac{H_{[1,1]}}{*S_{[1,1]}} &= A^{-20}q^{40} \left((q^{-16} - q^{-18} + 2q^{-22} - q^{-24} - q^{-26} + q^{-28})A^{-4} + \right. \\
&\quad + (-q^{-12} - q^{-14} - q^{-18} - 3q^{-20} - q^{-22} + q^{-24} - 2q^{-26} - 2q^{-28} - q^{-36} - q^{-38})A^{-2} + \\
&\quad + (2q^{-10} + q^{-12} + 2q^{-14} + 3q^{-16} + 4q^{-18} + 4q^{-20} + 4q^{-22} + 3q^{-24} + 6q^{-26} + \\
&\quad + 3q^{-28} + q^{-30} + 3q^{-32} + 4q^{-34} + 2q^{-36} + q^{-38} + q^{-40} + 2q^{-42} + q^{-44} + q^{-46}) + \\
&\quad + (-q^{-8} - 2q^{-10} - q^{-12} - 3q^{-14} - 5q^{-16} - 3q^{-18} - 5q^{-20} - 6q^{-22} - 5q^{-24} - 6q^{-26} - 4q^{-28} - 4q^{-30} - 6q^{-32} - \\
&\quad - 4q^{-34} - q^{-36} - 3q^{-38} - 5q^{-40} - 2q^{-42} - q^{-44} - 2q^{-46} - q^{-48} - q^{-50} - q^{-52})A^2 + \\
&\quad + (q^{-8} + q^{-12} + 2q^{-14} + q^{-16} + q^{-18} + 4q^{-20} + q^{-22} + 3q^{-24} + 3q^{-26} + q^{-28} + 3q^{-30} + \\
&\quad + 3q^{-32} + 2q^{-36} + 3q^{-38} + q^{-40} + 2q^{-44} + q^{-46} + q^{-48} + q^{-50} + q^{-56})A^4 \left. \right) = \\
&= \frac{A^{-20}q^{40}}{*S_{[1,1]}} \left(*S_{[3,3]} + *S_{[3,2,1]}(-q^{-6} + q^{-8} - q^{-10} + q^{-12} - q^{-14}) + *S_{[3,1,1,1]}q^{-20} + *S_{[2,2,2]}q^{-20} + \right. \\
&\quad + *S_{[2,2,1,1]}(q^{-14} - q^{-16} - q^{-18} + 2q^{-20} - 2q^{-24} + q^{-26} + q^{-28} - 2q^{-30} + q^{-32} - q^{-36} + q^{-38}) + \\
&\quad \left. + *S_{[2,1,1,1,1]}(-q^{-32} + q^{-36} - q^{-40} + q^{-44} - q^{-48}) + *S_{[1,1,1,1,1,1]}q^{-60} \right) \tag{337}
\end{aligned}$$

$$\begin{aligned}
\frac{H_{[2]}}{*S_{[2]}} &= A^{-20}q^{-40} \left((q^{28} - q^{26} - q^{24} + 2q^{22} - q^{18} + q^{16})A^{-4} + \right. \\
&+ (-q^{38} - q^{36} - 2q^{28} - 2q^{26} + q^{24} - q^{22} - 3q^{20} - q^{18} - q^{14} - q^{12})A^{-2} + \\
&+ (q^{46} + q^{44} + 2q^{42} + q^{40} + q^{38} + 2q^{36} + 4q^{34} + 3q^{32} + q^{30} + 3 \\
&+ q^{28} + 6q^{26} + 3q^{24} + 4q^{22} + 4q^{20} + 4q^{18} + 3q^{16} + 2q^{14} + q^{12} + 2q^{10}) + \\
&+ (-q^{52} - q^{50} - q^{48} - 2q^{46} - q^{44} - 2q^{42} - 5q^{40} - 3q^{38} - q^{36} - 4q^{34} - 6q^{32} - 4q^{30} - \\
&L - 4q^{28} - 6q^{26} - 5q^{24} - 6q^{22} - 5q^{20} - 3q^{18} - 5q^{16} - 3q^{14} - q^{12} - 2q^{10} - q^8)A^2 + \\
&+ (q^{56} + q^{50} + q^{48} + q^{46} + 2q^{44} + q^{40} + 3q^{38} + 2q^{36} + 3q^{32} + 3q^{30} + \\
&+ q^{28} + 3q^{26} + 3q^{24} + q^{22} + 4q^{20} + q^{18} + q^{16} + 2q^{14} + q^{12} + q^8)A^4 \Big) = \\
&= \frac{A^{-20}q^{-40}}{*S_{[2]}} \left(*S_{[6]}q^{60} + *S_{[5,1]}(-q^{48} + q^{44} - q^{40} + q^{36} - q^{32}) + \right. \\
&+ *S_{[4,2]}(q^{38} - q^{36} + q^{32} - 2q^{30} + q^{28} + q^{26} - 2q^{24} + 2q^{20} - q^{18} - q^{16} + q^{14}) + \\
&\left. + *S_{[4,1,1]}q^{20} + *S_{[3,3]}q^{20} + *S_{[3,2,1]}(-q^{14} + q^{12} - q^{10} + q^8 - q^6) + *S_{[2,2,2]} \right)
\end{aligned} \tag{338}$$

Alexander polynomials

$$\mathfrak{A}_{[1]} = q^8 - q^6 + 2q^2 - 3 + 2q^{-2} - q^{-6} + q^{-8} \tag{339}$$

$$\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = q^{16} - q^{12} + 2q^4 - 3 + 2q^{-4} - q^{-12} + q^{-16} \tag{340}$$

Jones polynomials

$$J_{[1]} = q^{-8} + q^{-12} - q^{-16} + q^{-18} - q^{-20} + q^{-22} - q^{-24} \tag{341}$$

$$J_{[1,1]} = 1 \tag{342}$$

$$\begin{aligned}
J_{[2]} &= q^{-16} + q^{-22} + q^{-28} - 2q^{-30} + 2q^{-34} - 2q^{-36} - q^{-38} + 3q^{-40} - 3q^{-44} + \\
&+ 2q^{-46} + q^{-48} - 4q^{-50} + 2q^{-52} + q^{-54} - 3q^{-56} + q^{-58} + 2q^{-60} - q^{-62} - q^{-64} + q^{-66}
\end{aligned} \tag{343}$$

Special polynomials

$$\mathfrak{H}_{[1]} = A^{-12} - 6A^{-10} + 6A^{-8} \tag{344}$$

$$\mathfrak{H}_{[1,1]} = \mathfrak{H}_{[2]} = (\mathfrak{H}_{[1]})^2 = A^{-24} - 12A^{-22} + 48A^{-20} - 72A^{-18} + 36A^{-16} \tag{345}$$

Ooguri-Vafa polynomials

$$\begin{aligned}
f_{[1,1]} &= \frac{\{A\}^2\{A/q\}\{Aq\}}{\{q\}} \left((q^{23} - q^{21} + q^{19} + q^{17} + q^{11} + q^7 + q^1)A^{-22} + \right. \\
&+ (-q^{27} - q^{23} - 2q^{21} - 2q^{19} - 2q^{17} - 3q^{15} - 4q^{13} - 4q^{11} - \\
&- 4q^9 - 3q^7 - 4q^5 - 4q^3 - 2q^1 - 2q^{-1} - 2q^{-3} - q^{-5} - q^{-7})A^{-20} + \\
&+ (q^{29} + 2q^{25} + 2q^{23} + 3q^{21} + 3q^{19} + 7q^{17} + 4q^{15} + 9q^{13} + 6q^{11} + 8q^9 + \\
&+ 8q^7 + 8q^5 + 5q^3 + 7q^1 + 4q^{-1} + 4q^{-3} + 2q^{-5} + 2q^{-7} + q^{-9} + q^{-11})A^{-18} \Big) \\
f_{[2]} &= \frac{\{A\}^2\{A/q\}\{Aq\}}{\{q\}} \left((-q^{-1} - q^{-7} - q^{-11} - q^{-17} - q^{-19} + q^{-21} - q^{-23})A^{-22} + \right. \\
&+ (q^7 + q^5 + 2q^3 + 2q^1 + 2q^{-1} + 4q^{-3} + 4q^{-5} + 3q^{-7} + 4q^{-9} + 4q^{-11} + \\
&+ 4q^{-13} + 3q^{-15} + 2q^{-17} + 2q^{-19} + 2q^{-21} + q^{-23} + q^{-27})A^{-20} + \\
&+ (-q^{11} - q^9 - 2q^7 - 2q^5 - 4q^3 - 4q^1 - 7q^{-1} - 5q^{-3} - 8q^{-5} - 8q^{-7} - 8q^{-9} - 6q^{-11} - \\
&- 9q^{-13} - 4q^{-15} - 7q^{-17} - 3q^{-19} - 3q^{-21} - 2q^{-23} - 2q^{-25} - q^{-29})A^{-18} \Big)
\end{aligned} \tag{346}$$

Special Ooguri-Vafa polynomials

$$\mathfrak{f}_{[2]} = -\mathfrak{f}_{[1,1]} = -\frac{(159A^8 - 360A^6 + 272A^4 - 76A^2 + 7)(A - A^{-1})^3}{A^{26}} \tag{348}$$

Numbers $N_{R,n,k}$

$N_{[1]} :$

$k \backslash n =$	-13	-11	-9	-7
0	-1	7	-12	6
1	-1	14	-34	21
2	0	7	-28	21
3	0	1	-9	8
4	0	0	-1	1

$N_{[1,1]} :$

$k \backslash n =$	-26	-24	-22	-20	-18	-16	-14
0	26	-271	1149	-2406	2644	-1467	325
1	283	-2908	12519	-26725	29712	-16483	3602
2	1231	-13215	59305	-130641	147643	-81894	17571
3	2846	-33702	161724	-372044	429324	-237978	49830
4	3939	-54000	283511	-689018	815314	-451252	91506
5	3445	-57565	338189	-878533	1070393	-590788	114859
6	1939	-42074	283375	-797042	1004073	-551784	101513
7	697	-21353	169548	-524186	685766	-374583	64111
8	154	-7507	72733	-251892	343856	-186357	29013
9	19	-1792	22182	-88275	126395	-67847	9318
10	1	-277	4692	-22269	33635	-17852	2070
11	0	-25	654	-3931	6302	-3302	302
12	0	-1	54	-460	788	-407	26
13	0	0	2	-32	59	-30	1
14	0	0	0	-1	2	-1	0

$N_{[2]} :$

$k \backslash n =$	-26	-24	-22	-20	-18	-16	-14
0	31	-333	1434	-3026	3339	-1857	412
1	396	-4107	17757	-37965	42249	-23472	5142
2	2031	-21561	95976	-210260	237023	-131551	28342
3	5548	-63635	299280	-680083	780367	-432718	91241
4	9162	-118624	602538	-1436213	1684844	-933053	191346
5	9738	-148552	831465	-2101076	2529808	-1397916	276533
6	6853	-129374	814438	-2205943	2736313	-1506797	284510
7	3212	-79780	577794	-1698237	2178428	-1193693	212276
8	988	-35018	299618	-969806	1291661	-703214	115771
9	191	-10859	113471	-412141	572417	-309146	46067
10	21	-2323	31010	-129656	188672	-100927	13203
11	1	-326	5950	-29730	45559	-24104	2650
12	0	-27	760	-4821	7823	-4088	353
13	0	-1	58	-523	904	-466	28
14	0	0	2	-34	63	-32	1
15	0	0	0	-1	2	-1	0